

Quarter Wit, Quarter Wisdom- Veritas Prep Karishma Blog Post's Collection

1. Writing Factors of ugly numbers

Let's start with a very interesting and important topic that GMAT loves to test you on – Factors.

Factors are the divisors of a number. 1 is a factor of every number and the number itself is also a factor of that number. For the sake of giving the definition, let me say: A positive integer x is a factor of a positive integer N when there exists another positive integer y such that $x \times y = N$. In other words, when N is divided by x , it doesn't leave any remainder.

Now, let's cut to the chase and go on to real business.

Tell me, how many factors does 315 have and what are they? (Isn't that an ugly number! Unlike 36 or 72 or 81 – numbers easy to work with. But let's work with what we have.)

First of all, let me break it down to smaller numbers: $315 = 63 \times 5 = 7 \times 9 \times 5$ (looks better now)

Now we know that 5, 7, 9, 3 are all factors of 315 because they divide 315 completely. But there are others too e.g. 15 (15×21 gives 315). So how do we ensure that we get ALL the factors of 315?

Let me digress here with a few questions:

Q: Is 2 a factor of 315?

A: No! 2 does not divide 315 completely.

Q: Is 9 a factor of 315?

A: Yes, we can see that it is. 9×35 gives us 315.

Q: So then 35 is also a factor of 315, isn't it?

A: Yes, because 35 will divide 315 completely and give 9 as quotient.

Similarly, if 7 is a factor of 315, we should have a corresponding factor 45 such that, $7 \times 45 = 315$. Get the picture? (Let's assume you nodded your head.)

Back to the topic at hand now. Let's retain the Q&A format.

Q: 315 has lots of factors: 1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315. How do we get all of them?

A: For that, I first need to break it down to its prime factors $315 = 3^2 \times 5 \times 7$. We get all the factors of 315 by combining the prime factors in as many different ways as we can. i.e. we take a 3 alone; it is a factor of 315. We take a 3^2 alone; it is also a factor of 315. We take a 5 alone; it is again a factor of 315. We take a 3 and a 5 and multiply them to get 15 as a factor and so on...

Q: In how many different ways can we combine the prime factors of 315?

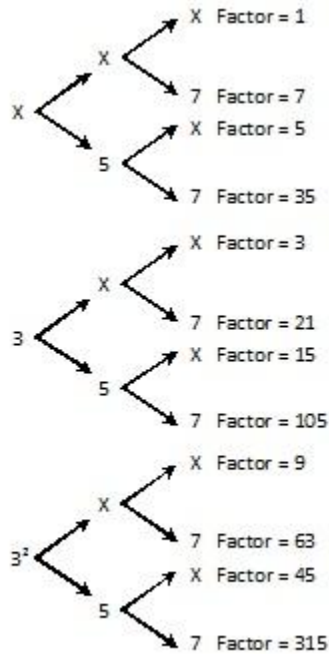
A: In $3 \times 2 \times 2$ ways

Q: Wait a sec! How did we get this?

A: We have to take a combination of 3, 3, 5 and 7 to make a factor. We can do this in any way we like. We can

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make a factor by taking a 3, a 5 and a 7. We get 105. Or we can make a factor by taking two 3s, no 5 and a 7. We get 63 as a factor. Therefore, we can choose 3 in three ways (take no 3, take one 3 or take two 3s), 5 in two ways (no 5 and one 5) and 7 in two ways (no 7 and one 7). So there are, in all, $3 \times 2 \times 2 = 12$ ways to make a factor of 315. The diagram below illustrates this concept.



In general, if a number can be written as $N = x^p \cdot y^q \cdot z^r \dots$ (where x, y, z are all distinct prime numbers), then total number of factors of N is $(p + 1)(q + 1)(r + 1) \dots$

(the $+ 1$ indicates the case in which you do not take that prime factor while making your factor)

Let me write down all the factors of 315 in increasing order:

1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315

Q: Do you notice a pattern? No?

Try again now

1, 3, 5, 7, 9, 15,

315, 105, 63, 45, 35, 21

Q: How about it?

A: Look at each column individually: $1 \times 315 = 315$; $3 \times 105 = 315$; $5 \times 63 = 315$; $7 \times 45 = 315$; $9 \times 35 = 315$; $15 \times 21 = 315$!

Q: Co-incidence?

A: I think not. Numbers equidistant from the left and the right multiply to give the original number. This is where the importance of the definition of factors given above comes in (You thought I was just wasting time by giving the definition, isn't it?) Every factor x (let's say 7) of a number (315 here) will have a factor y (45 here) such that $x \times y = N$ ($7 \times 45 = 315$). The smaller the x , the greater the y to make up N . So if a factor is third smallest, it will multiply the

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third greatest to give you the number N.

To now, if we want to write down all the factors of 315, we just write down the small ones

1, 3, 5, 7, 9, 15

And then next to them write their corresponding big factors:

1, 3, 5, 7, 9, 15

315, 105, 63, 45, 35, 21

To ensure that we haven't missed any of the small ones, we can use the formula and find out how many total factors does the number have. Life made easy!

Something to think about:

Q: What happens if the total number of factors is odd?

2. Factors of perfect Squares

What happens if the total number of factors of a number is odd?

Let us take the example of $N = 100$. Break it down into its prime factors.

$$100 = 10 \times 10 = 2^2 \times 5^2$$

How many factors will it have? $(2 + 1)(2 + 1) = 9$. Let us write them down:

1, 2, 4, 5, 10, 20, 25, 50, 100

$$1 \times 100 = 100; 2 \times 50 = 100; 4 \times 25 = 100; 5 \times 20 = 100.$$

10 is in the middle. It has no companion with which it could multiply and give 100! But, 10 will multiply with itself to give 100 ($10 \times 10 = 100$). This means, we have a factor which multiplies by itself to give the number. Hence the number N (=100 here) must be a perfect square!

ONLY perfect squares will have odd number of total factors and ALL perfect squares will have an odd number of total factors.

Referring back to the formula of total factors, since each of the powers is even, so each of (power + 1) is odd; their product will definitely always be odd.

A short Q&A session here:

Q: How many odd factors does 100 have?

A: 3 (1, 5 and 25 – An odd number)

Q: How many even factors does 100 have?

A: 6 (2, 4, 10, 20, 50, 100 – An even number)

Q: Is this just a co-incidence?

A: No. Let me explain:

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$$100 = 2^2 \times 5^2$$

If we want to find out its odd factors, we need to find how many factors we can make out of 5^2 . (We need to drop the 2s since 2s create even factors.) We can make only $(2 + 1) = 3$ odd factors. Since power of 5 is even, (power + 1) will be odd. **A perfect square always has odd number of odd factors.**

If we want to find out its even factors, we multiply each of the odd factors by 2 or 2^2 . We can take 2 in two ways (one 2 or two 2s). We cannot take no 2 because that just leaves us with an odd factor. So we get $3 \times 2 = 6$ even factors. **A perfect square always has even number of even factors.**

What happens when we add all these factors together? Adding the odd number of factors (3) with the even number of factors (6) will give us an odd number of total factors (9). This will be true for all perfect squares.

Let's look at another example.

$$N = 2^4 \times 7^2 \times 11^6$$

This is a perfect square because powers of all prime factors are even.

$$\text{Total number of factors} = (4 + 1)(2 + 1)(6 + 1) = 105$$

$$\text{Total number of odd factors} = (2 + 1)(6 + 1) = 21 \text{ (Dropped the 2)}$$

$$\text{Total number of even factors} = 21 \times 4 = 84 \text{ (Each odd factor multiplied by } 2/2^2/2^3/2^4)$$

Or if you like, we can say out of a total of $(4 + 1)(2 + 1)(6 + 1)$ factors, $4(2 + 1)(6 + 1)$ are even (since they have a 2) and $1(2 + 1)(6 + 1)$ are odd since they do not have a 2.

Sum of the 21 odd factors will be odd and sum of the 84 even factors will be even. Hence, sum of all the factors will be odd.

Note: The sum of all factors of a perfect square is always odd but if the sum of all factors of a number is odd, we cannot say that it must be a perfect square. E.g. 18 has 6 factors (1, 2, 3, 6, 9, 18). Their sum is 39, an odd number but 18 is not a perfect square.

A Great Little Application: If we want to find whether 91 is a prime number, can the logic of factors help us? (It must since it is a part of the 'Factors' post.)

A number is prime when it has no factors other than 1 and itself. So if we want to find out whether 91 is prime, we need to find if it has any other factors. I can't think of what to break it down into so my first impulse is to say that it is prime. But let me check to be sure that we are not missing anything. 91 is not divisible by 2, not by 3 either, not by 5, but it is divisible by 7! So it is not prime.

What about 83?

Let me check – not divisible by 2, not by 3, not by 5, not by 7, not by 11.... How long do we need to go? Do we need to check for all prime numbers till 83? No! Because we saw above that factors occur in pairs. If a big number is a factor of 83, there will be a corresponding small number which will also be a factor of 83. If we couldn't find any factor of 83 in the small numbers, we wouldn't find any in the big numbers too... Now, 'small' and 'big' are arbitrary terms. We want to know exactly till what number should we check. Do you remember which number was exactly in the middle of the factors? Yes, the square root! Even if the square root isn't a factor like in case of 315, if we do place it among the factors, it will be in the middle. E.g. square root of 315 is around 17.7 (because square of 17 is 289 and square of 18 is 324. If we place 17.7 among its factors, it will come after 15 but before 21 i.e. right in the middle.

1, 3, 5, 7, 9, 15, 17.7, 21, 35, 45, 63, 105, 315

So we should check till the square root of the number.

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Since square root of 83 is a little more than 9, we should check for all prime numbers less than 9. If there isn't any factor in those, there wouldn't be any factor in the numbers greater than 9.

Quick Question: Is 103 prime?

Not divisible by 2, not by 3, not by 5, not by 7... that's it. It is prime.

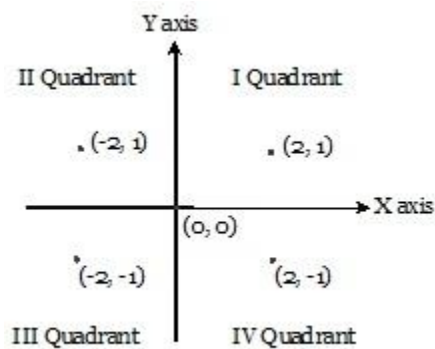
(Square root of 103 is a little more than 10 so we only need to check for primes less than 10.)

3. Bagging the Graphs-I

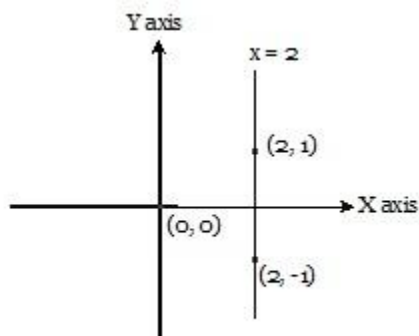
One thing that I would like to suggest to increase speed in Arithmetic is Multiplication Tables. Much to my dad's chagrin, I am still a little lost when confronted with 16×7 or 17×8 or 18×7 (I know the last two are 126 and 136 but in what order, I am not sure) but rest I can pretty much manage. And many a times, while solving little toughies, I have blessed my dad for his incessant reproach regarding tables in days yonder.

Now, the one thing that I would like to suggest to increase speed in Coordinate Geometry and Algebra is learning how to draw graphs. Know how to draw a line from its equation in under ten seconds and you shall solve the related question in under a minute. For now, take my word for it and go ahead.

This is what the xy coordinate axis looks like:



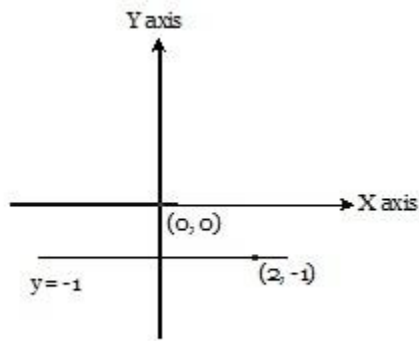
If we were to draw the line $x = 2$ this is what it would look like:



On this line, at every point, x coordinate is 2. y coordinate varies from point to point.

Similarly, $y = -1$ is as shown:

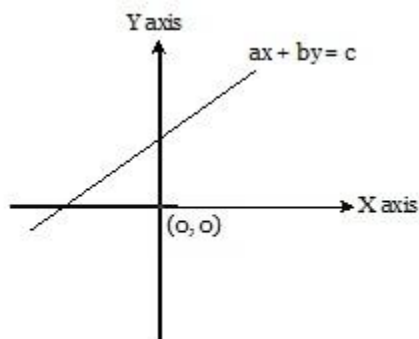
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Taking a cue from above, can you tell me, how you would draw $y = 0$? Of course it is the X-axis. Where is $y = 0$? At every point of the X axis!

And then the equation of y axis must be ... yes, $x = 0$.

But usually the kind of lines we need to draw, look something like this:

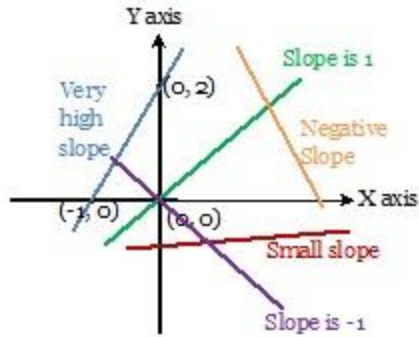


Any given line on the xy plane can be uniquely described using two characteristics – the line's slope and a point through which the line passes.

Slope of a line:

The slope of the line is just a measure of how tilted it is. As x increases, if y increases much more, the line becomes more tilted. Look at the blue line below. When x increased by 1 (from -1 to 0), y increased by 2 (from 0 to 2) so the line has a slope of 2. The green line below has a slope of 1. When x increases by 1 unit, y also increases by 1 unit. On the other hand, the red line has a very small slope. When x increases by 1 unit, y increases by very little. So what about the orange line? There, when x increases, y decreases! Of course the slope is negative there. The purple line has a slope of -1. When x increases by 1 unit, y decreases by 1 unit. So we see that when you go from left to right (à), if the line is going up, its slope is positive; if it is going down, its slope is negative.

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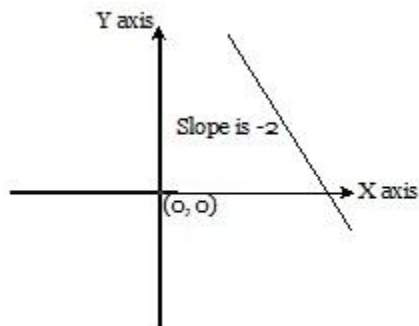
A Point on the line:

The second characteristic that defines a line is a point through which it passes. This could be expressed in many ways: y intercept, x intercept or (x, y)

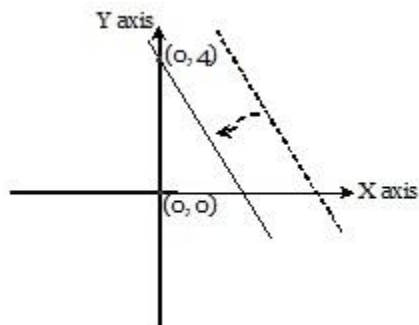
When I say y intercept of a line is 4, it just means that it passes through $(0, 4)$ i.e. it cuts the y axis at point 4. When I say the x intercept of a line is -2, it just means the line passes through $(-2, 0)$ i.e. it cuts the x axis at point -2. Or I could simply say that the line passes through $(1, 6)$. If I have any one of these and the slope, I can draw a unique line. Let's try it.

Given that the slope of a line is -2 and its y intercept is 4, how will you draw the line?

First of all, since slope is -2, the line will look something like this



Now, since it cuts the y axis at 4, the line can be drawn like this:



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So do the questions tell you that slope is -2 and y intercept is 4? At GMAC, they don't really like you that much! What they generally give is the equation of a line, say $2x + y - 4 = 0$. You will re-arrange this equation to get $y = -2x + 4$ (Remember $y = mx + b$ where m is the slope and b is the y intercept?). You get -2 as the slope and 4 as the y intercept.

Any equation can be put in this format.

$3x + 4y - 6 = 0$. Re-arrange to get $y = -3x/4 + 6/4$. Slope = $-3/4$ and y intercept = $3/2$.

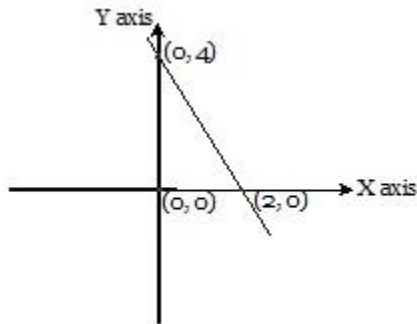
Soon, we will discuss another quick method of drawing a line given its equation.

4. Bagging the Graphs-II

Would you say it is easy breezy to draw a line if two points through which it passes are given? Sure. Plot the points approximately and join them! There you have your required line.

A quick method of drawing the line represented by an equation: find two points through which it passes, plot the points and join them.

The easiest points to find are x and y intercepts. Let's say I have the equation $2x + y - 4 = 0$. In this equation, if I put $x = 0$, I get $y = 4$ (the y intercept). This means the point $(0, 4)$ lies on the line. If I put $y = 0$, I get $x = 2$ (the x intercept). So the point $(2, 0)$ lies on the line too. Plot them and join them.



Are you wondering why I worked with the slope first when this easy method existed? Because slope method has its own utility and will come in very handy in some questions. Anyway, I had a nagging feeling that if I suggested this approach first, you may not bother about the slope concept at all!

One data sufficiency question to bind it all:

Question: Every point in the xy plane satisfying the condition $ax + by - c = 0$ is said to be in region R . If a , b and c are real numbers, does any point of region R lie in the third quadrant?

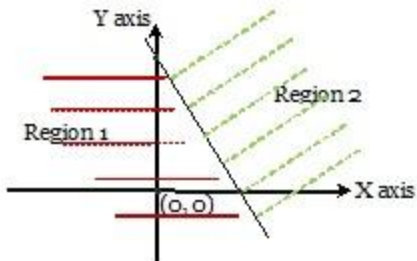
Statements:

1. Slope of the line represented by $ax + by - c = 0$ is 2.
2. The line represented by $ax + by - c = 0$ passes through $(-3, 0)$.

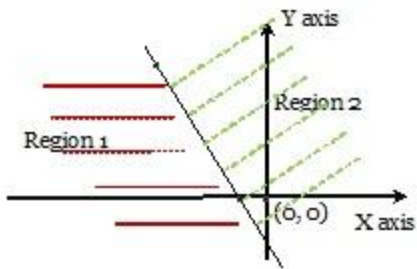
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Solution:

First of all, notice that $ax + by - c = 0$ or $ax + by = c$ is the equation of the same line. A line divides the plane into two regions. One of them, where every point (x, y) satisfies $ax + by > c$, is region R. As of now, we do not know what the line looks like and which of the given two regions is region R.

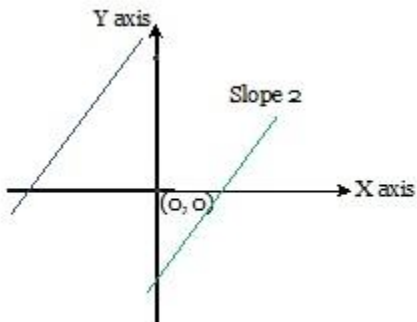


If the line is as shown above, both the regions will have points in the first, second and fourth quadrants but only Region 1 will have points in the third quadrant.



But if the line is as shown in this figure, both the regions will have points in the second, third and fourth quadrants but only Region 2 will have points in the first quadrant. There are many different cases. We see that if a line passes through a particular quadrant, both regions lie in that quadrant. Else, only one of the two regions lies in that quadrant. For example, in the figure above, the line passes through second, third and fourth quadrants and hence, both regions lie in these quadrants. But it doesn't pass through quadrant 1 and hence only Region 2 lies in the first quadrant. Let's go on to the statements now.

Statement 1: Slope of the line represented by $ax + by - c = 0$ is 2.

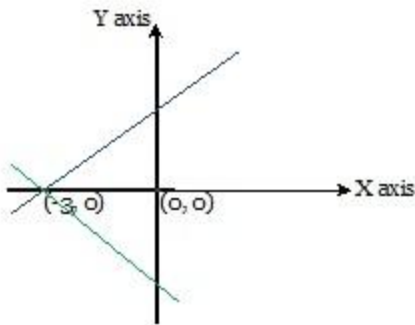


If the slope of the line is 2, it

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will always pass through the third quadrant and hence both regions will have points in the third quadrant. (Note that a line with a positive slope will definitely pass through the first and the third quadrants.) Statement 1 is sufficient alone.

Statement 2: The line passes through $(-3, 0)$.



A line passing through $(-3, 0)$ could be the blue line or the green line. In either case, the line will pass through the third quadrant and hence, will have both regions in the third quadrant. So it is sufficient too? What about the x axis? That is also a line passing through $(-3, 0)$. But it does not pass through the third quadrant. We would need the equation of the line to find out whether our region R lies in the third quadrant. The equation of x axis is $y = 0$. So the required region R is given by $y > 0$ i.e. the first and second quadrants. Here region R does not have any points in the third quadrant. Using just the information given in Statement 2, we cannot say whether a point of region R lies in the third quadrant or not. Statement 2 alone is not sufficient.

Since statement 1 alone is sufficient and statement 2 alone is not, answer is (A).

5. Bagging the Graphs-III

While introducing this concept in Part I of Graphs, I had mentioned: “the one thing that I would suggest to increase speed in Co-ordinate Geometry and Algebra is Graphs”

Did you wonder why I included “Algebra” here? If yes, then this post will answer your question. In part 2, I gave an example of a Geometry question that can be easily solved using Graphs. In this post, I will take up an Algebra question for which you can do the same.

In part I of Graphs, I had also mentioned “Learn how to draw a line from its equation in under ten seconds and you shall solve the related question in under a minute.” After this post, you won’t have to take my word for it!

Before I begin, let me also add here that this Data Sufficiency question is not a question I created. So don’t think I made it to conveniently suit my needs. It is a question someone asked me on a GMAT forum and was created by some third party. I chose it to demonstrate the beauty of graphs to you.

Question: If x and y are positive, is $4x > 3y$?

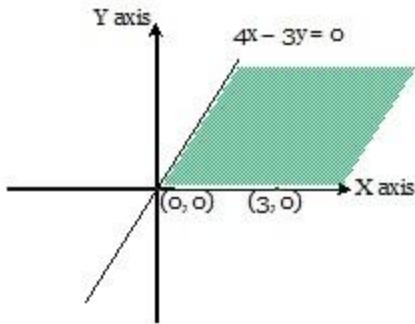
Statements:

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1. $x > y - x$
2. $x/y < 1$

Let us look at the question stem first: x and y are positive, 'is $4x > 3y$?' or rephrase it as 'is $4x - 3y > 0$?'

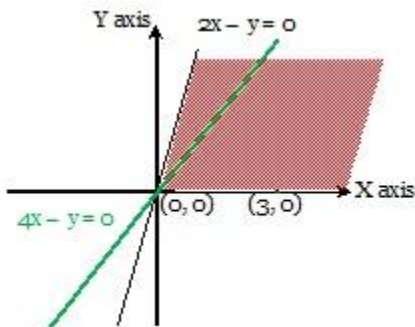
Let us draw $4x - 3y = 0$. Then we can figure out which region represents $4x - 3y > 0$. When $x = 0$, $y = 0$ so the line passes through $(0, 0)$. The slope of the line is $4/3$. This is what it looks like:



The line has divided the graph into two regions: $4x - 3y < 0$ and $4x - 3y > 0$. Let us check in which region, the point $(3, 0)$ lies. (This is an arbitrary choice. You can check for any point.) When you put $x = 3$ and $y = 0$ in $4x - 3y > 0$, you get $12 > 0$ which is true. Hence the point $(3, 0)$ lies in $4x - 3y > 0$ region. Since x and y are positive, we are only concerned with quadrant I. The shaded Green region is where $4x - 3y > 0$. So the question boils down to: "Does the point (x, y) lie in the shaded Green region for all values of x and y ?"

Statement 1: $x > y - x$ or $2x - y > 0$

Draw $2x - y = 0$. When $x = 0$, $y = 0$ so this line passes through the center. The slope of the line is 2. The slope of this line, 2, is greater than $4/3$, the slope of the line drawn above. Hence, this line will be steeper than the line drawn above. Check for point $(3, 0)$ again to find whether it lies in region $2x - y > 0$. Putting $x = 3$ and $y = 0$, we get $6 > 0$ which is true so the region $2x - y > 0$ includes the point $(3, 0)$ and is as shown below:



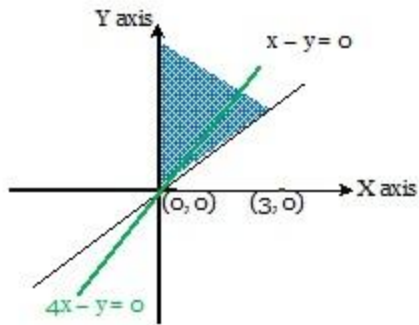
The Red shaded region here includes all the points of the Green shaded region above plus some more. Hence all points of Red region may not lie in the Green region. Therefore, if values of x and y satisfy $2x - y > 0$, they may or may not satisfy $4x -$

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$3y > 0$. Hence statement 1 alone is not sufficient.

Statement 2: $x/y < 1$

Since x and y are positive, we can multiply both sides of the inequality by y to get $x < y$ or $x - y < 0$. Draw $x - y = 0$. When $x = 0$, $y = 0$ so this line passes through the center. The slope of the line is 1. The slope of this line, 1, is less than $4/3$, the slope of the line in the question. Hence, this line will be less steep than the line in the question stem. Check for point $(3, 0)$ again to find whether it lies in region $x - y < 0$. Putting $x = 3$ and $y = 0$, we get $3 < 0$ which is not true so the region $x - y < 0$ does not include the point $(3, 0)$ and is as shown below:

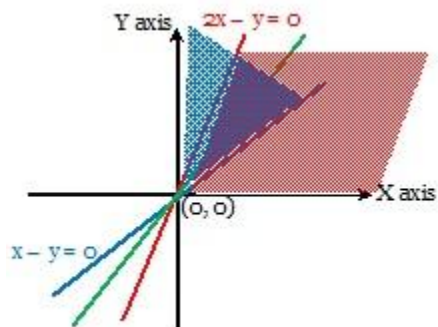


Note: Our concern is limited to

first quadrant since x and y are both positive.

The Blue shaded region here includes some points of the Green shaded region above plus some more. Hence all points of Blue region may not lie in the Green region. Therefore, if values of x and y satisfy $x - y < 0$, they may or may not satisfy $4x - 3y > 0$. Hence statement 2 alone is not sufficient.

Taking both statements together, x and y will have values that overlap in the Red and the Blue region as shown in the graph below. Some of these values will lie in the Green shaded region above, some will not.



So even if we take both

statements together, they are not sufficient. Answer (E).

I hope you have come to appreciate the wide range of applicability of graphs. Next time, I will introduce a graphical way of working with Modulus and Inequalities.

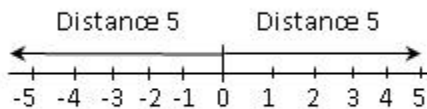
6. Do what Dumbledore Did.

I can say with reasonable certainty that until and unless you are a quant jock, you hate Inequalities (especially when they present themselves in DS questions). I can also say the same thing about Modulus (perhaps, not to the same extent!). So I can imagine what you feel when you come across a DS question with both Modulus and Inequalities! I am also certain that a small part of you, a very small part indeed, is secretly thrilled to see such a question because it implies that you are doing well in the exam and the software is getting jittery and trying to give you harder and harder questions. But wouldn't it be something if you could crack the question in a minute and then say, "What else you got?"

It is not very difficult to do that, in my opinion. GMAT is not about solving complicated equations/inequalities. It is about using your bare fundamentals, quietly, inconspicuously... I am reminded of Harry Potter and the Half-Blood Prince here (Yes, I am a Harry Potter fan and this time, I cannot blame my little daughter for it!). Dumbledore and Harry go to some cave to get a Horcrux and Dumbledore quietly works his way through, feeling around for magic. At that point, the author says, "He had never seen a wizard work things out like this, simply by looking and touching; but Harry had long since learned that bangs and smoke were more often the marks of ineptitude than expertise." I think it is quite an appropriate analogy. GMAT is not about booms and bangs and fancy equations; it is more ethereal; you have to be more in touch with your concepts...

Anyway, I don't want you to run away if you came here looking for some solid fundamentals. Let me start by saying that inequality just implies less than or greater than. Don't try to read too much into it. Modulus stands for the distance of a point from 0 on the number line. When I say $|x| = 5$, it means x is a point at a distance 5 from 0 on the number line. Take a minute to go through it again: x is a point at a distance 5 from 0 on the number line. Which points are at a distance 5 from 0 on the number line?

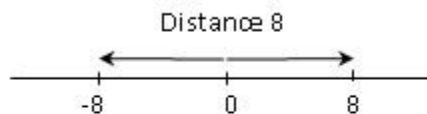
$$|x| = 5$$



So $x = 5$ or -5

Points 5 and -5. So x can take two values: 5 or -5. Similarly, if $|x| = 8$, x can take two values $x = 8$ or $x = -8$.

$$|x| = 8$$



So $x = 8$ or -8

$$|x| = -2$$

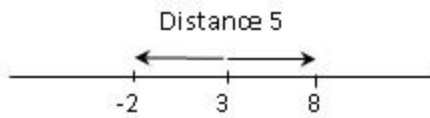
Not possible because distance cannot be negative

Now, what if you have something like $|x-3| = 5$? What does this mean? The distance you are looking for is still 5. The only difference is that the distance is from the point 3 now.

Let's look at the figure to see what values x can take.

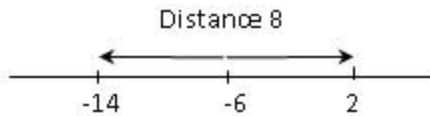
Quarter Wit_Quarter Wisdom- Part-1

$$|x-3| = 5$$



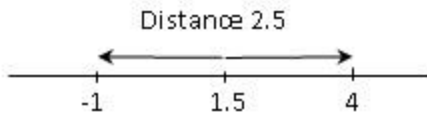
So $x = 8$ or -2

$$|x+6| = 8$$



So $x = 2$ or -14

$$|2x-3| = 5 \rightarrow 2|x-1.5| = 5 \rightarrow |x-1.5| = 2.5$$



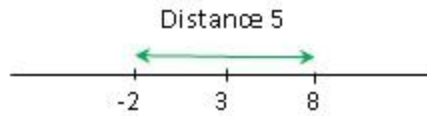
So $x = -1$ or $x = 4$

In the last example above, we say that twice the distance from 1.5 is 5. So we want the point that is 2.5 away from 1.5.

If this makes sense to you, we could include an inequality sign here. Things will still be no different. For example, what values can x take if $|x-3| < 5$. We need those points where distance from point 3 is *less than* 5. At all the points depicted by the green line in the diagram below, the distance from 3 is less than 5. Point 4 is at a distance 1 away from 3, point 7 is at a distance 4 away from 3, 0 is at a distance 3 away from 3 and so on... Point 8 is at a distance 5 away from 3 so all points lying between 3 and 8 are at a distance less than 5 from 3. Point 3 is at a distance 0 from 3 so it is also included but 8 is not since the inequality doesn't have an equal to sign.

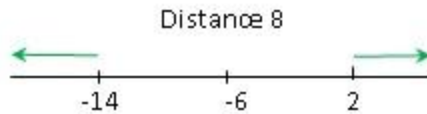
Quarter Wit_Quarter Wisdom- Part-1

$$|x-3| < 5$$



$$\text{So } -2 < x < 8$$

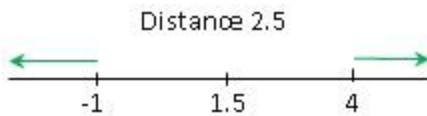
$$|x+6| \geq 8$$



Distance from -6 should be greater than 8.

$$\text{So } x \geq 2 \text{ or } x \leq -14$$

$$|2x-3| > 5 \rightarrow |x-1.5| > 2.5$$



$$\text{So } x < -1 \text{ or } x > 4$$

This is an extremely efficient way of working with inequalities + modulus.

7. Of Pounds & Ounces

The other day, I stepped on the weighing scale after a long time (which included a 10-day-pigging-out vacation so I was a little apprehensive in the first place). When I saw the figure displayed, my head spun for a minute. What the...? How is it possible? I got down from the scale, tapped it here and there a bit — willing it to be sensible — and with my heart drumming in my ears, got on again. Still the same darned figure! Really now!

And then it struck me. It was showing me my weight in pounds instead of the usual kilograms I like to see it in (weight in double digits is so much more reassuring!). The constriction in my chest relaxed and I breathed again. I remembered I had converted it to pounds to check the weight of my check-in luggage before leaving for my vacation. I have half a mind to sue those weighing scale manufacturers — the confusion caused by offering the feature of both units is hazardous to the health of people, an outcome which is in direct contrast to the intended use of the instrument.

What happened with me is just one example of the tons of heartaches caused by multiple units used. But since there is no running away from the reality that various units of measurement are used for the same physical quantity, you will occasionally need to convert one to the other. Let us learn how to do it quickly and efficiently through the use of some examples similar to GMAT questions.

Example 1: Jodie runs her Prius 90 miles in two hours. In how many seconds, does she cover 132 feet? (Given 1 mile = 5280 feet)

Quarter Wit_Quarter Wisdom- Part-1

Solution: The solution involves two steps: First one makes the units consistent and second one solves for time. We know Jodie covers 90 miles in 2 hours so her speed is 45 mph. We need to convert her speed from miles/hour to feet/second because distance is given in feet and the answer needs the time in seconds.

$$\text{Speed} = \frac{45 \text{ miles}}{1 \text{ hour}} = \frac{45 \times 5280 \text{ feet}}{3600 \text{ seconds}} = 66 \text{ feet/second}$$

Replace a mile by 5280 feet (because 1 mile = 5280 feet) and 1 hour by 3600 seconds (1 hour has $60 \times 60 = 3600$ seconds) in the expression. We get Jodie's speed in feet/sec. Now, to solve the question, we simply use the Time-Speed-Distance formula which is,

$$\text{Time Taken} = \frac{\text{Distance}}{\text{Speed}} = \frac{132 \text{ feet}}{66 \text{ feet/sec}} = 2 \text{ sec}$$

The solution to the above problem comes naturally to most people. They may not follow these same steps but nevertheless, they do not find it intimidating. In that respect, I haven't added any value yet. But what is important here is the procedure because it can be helpful in solving problems that are a little spooky for some. Let me explain by taking a trickier example:

Example 2: Julia drives at a speed of 132 feet/sec. How many miles away is she from her home, if it will take her 2 hours to reach home? (Given 1 mile = 5280 feet)

Solution: Here we have Julia's speed in feet/sec and we need to convert it to miles/hour because the distance asked is in miles and the time taken is in hours.

Given 1 mile = 5280 feet, it follows that 1 foot = $1/5280$ miles.

We know that 1 hour = 3600 seconds, which implies that 1 second = $1/3600$ hours.

Making the desired replacements in the expression, we get:

$$\text{Julia's Speed} = \frac{132 \text{ feet}}{1 \text{ sec}} = \frac{132 \times 1/5280 \text{ miles}}{1/3600 \text{ hour}} = 90 \text{ miles/hour}$$

Replace a foot by $1/5280$ miles and 1 sec by $1/3600$ hour (Same procedure as in the solution above.)

Using the Time-Speed-Distance formula, we get

$$\text{Distance} = \text{Speed} \times \text{Time} = 90 \text{ miles/hr} \times 2 \text{ hrs} = 180 \text{ miles}$$

There's your answer!

Don't get lost in the conversion. This simple one-step method will not let you down.

8. The Holistic Approach to MODS

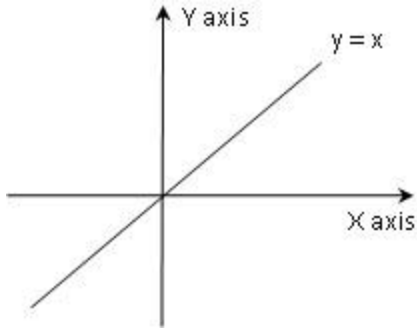
I have a dream... A dream that one day, I will see my students making a bonfire of all their pens and pencils... that I will see them lost in thought in my Quant class, occasionally drawing lines and curves on a drawing sheet with colorful crayons... I will see them coming up with innovative logical solutions, just like that... But I know that no dream of mine is realized until and unless I keep my nose to the grindstone (I am not waiting with bated breath to achieve that elusive target weight.) So on this particular sleepless night, I will write a post with some more figures, figures that make complicated questions look like easy pickings. Let me explain using step by step approach.

A Complicated Question: If $y = ||x - 5| - 10|$, for how many values of x is $y = 1$? (I remember once someone said, "I think I would rather eat spinach than try such questions.")

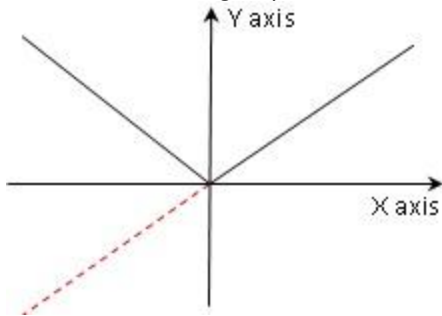
Easy Pickings:

Graph of $y = x$ is a line passing through the center with *slope 1*.

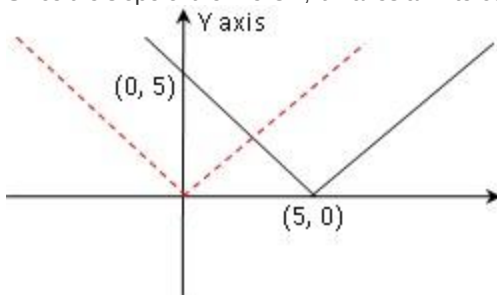
Quarter Wit_Quarter Wisdom- Part-1



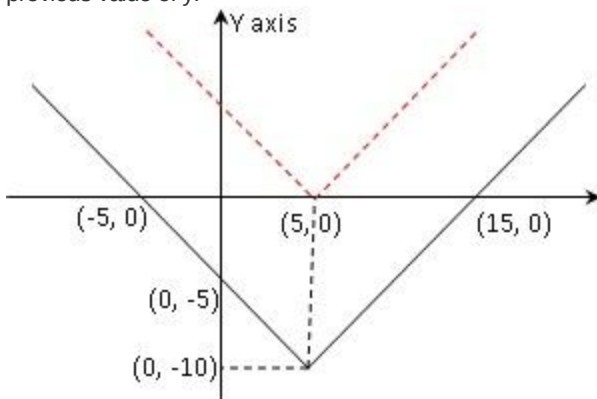
Graph of $y = |x|$ is as shown below. Modulus cannot be negative so all negative values of y are flipped to positive. (The red line shows the original position for reference.)



To get the graph of $y = |x - 5|$, shift the above graph 5 units to the right on the x axis. This is so because in the graph above, $y = 0$ when $x = 0$. But in the required graph, y should be 0 when $x = 5$. Hence the point at $(0, 0)$ shifts to $(5, 0)$. Since the slope of the line is 1, it makes an intercept of 5 on the y axis.

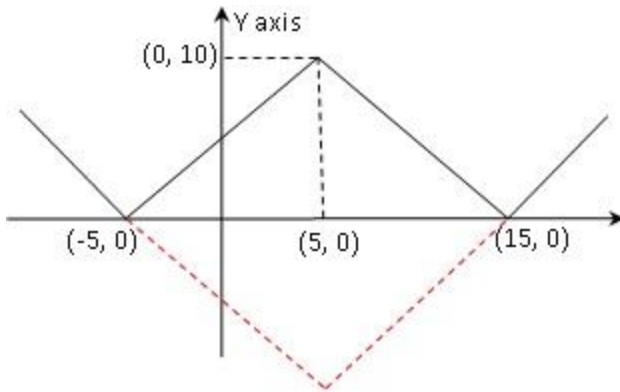


The graph of $y = |x - 5| - 10$ is just the above graph shifted down by 10 units because now y is 10 less than every previous value of y .

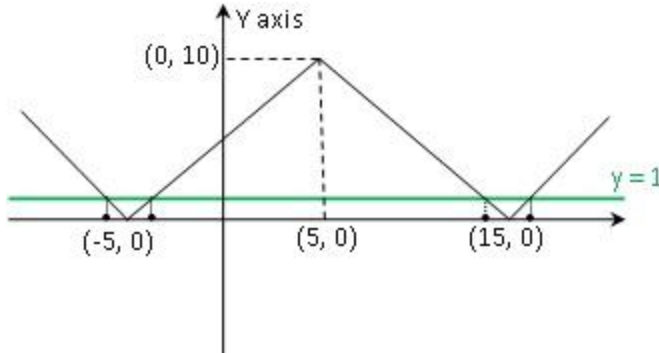


Now we need to take the modulus of the equation above to get $y = ||x - 5| - 10|$. Since a modulus is never negative, whatever part of the graph is below the x axis i.e. in quadrants III and IV, gets reflected above the x axis in quadrants I and II.

Quarter Wit_Quarter Wisdom- Part-1



This is the graph we wanted. We see that the line $y = 1$ (shown in green below) intersects this graph at 4 points. So $y = 1$ when $x = -6$ or -4 or 14 or 16 .



Put these values in the given equation if you want to cross check. Once you get the hang of it, you can arrive at this graph in under a minute! Such tricky questions can be elegantly handled using this approach. In fact, we can add many more levels of complexity and still easily arrive at our answer. For shakes, try out the graphs of $y = |||x - 5| - 10| - 5|$ and $y = |||x| - 3| - x|$!

Now that you have lost your sleep, I think I will sleep easier!

9. The Holistic Approach to MODS-II

There is one more important concept in Modulus that I want to discuss. Once it is done, we will bid farewell to Mods (for the time being at least), I promise. The concept involves dealing with multiple Mod terms (I will explain in just a minute). Before I start with the discussion, let me point out that it is relevant only if you are looking for a 50/51 in Quant and if you are looking for a 50/51 in Quant, then it is definitely relevant (I remember seeing a mean Modulus question in my GMAT a while back). But remember, don't waste time on advanced Modulus questions if you are uncomfortable with Number Properties or other such high-weightage topics. Only when you are above 48 consistently in Quant, should you spend time on the two posts titled 'Holistic Approach to Mods'. That said, everyone is welcome to read the posts and elicit his/her own takeaways.

Let's start.

Our problem for today is:

For what value of x , is $|x - 3| + |x + 1| + |x| = 10$?

The one good thing about GMAT is that it gives you five options. Here, the five options will be the possible values of x . So obviously, we will not waste time solving this question. We will just plug in the values and find out which value gives the sum 10. So let me write the complete question here:

Quarter Wit_Quarter Wisdom- Part-1

For what value of x , is $|x - 3| + |x + 1| + |x| = 10$?

- (A) 0
- (B) 3
- (C) -3
- (D) 4
- (E) -2

When you put $x = 4$, you get $|4 - 3| + |4 + 1| + |4| = 10$. So answer is (D).

When I made this question and was putting the options, obviously I had to solve the question to find the value of x (one option has to be the correct answer after all!) Using the method I will just discuss, I did it in my mind in under a minute! Curiosity piqued? I hope so.

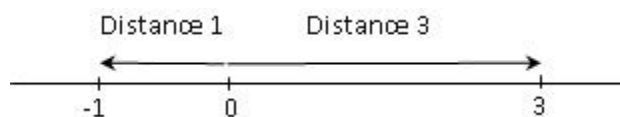
Let us change the question a little to rope you in.

For how many values of x , is $|x - 3| + |x + 1| + |x| = 10$?

- (A) 0
- (B) 1
- (C) 2
- (D) 3
- (E) Infinite

(Previously [we discussed](#) that Modulus represents distance. In this post, we will build upon that concept.)

In simple language, the question tells us that x is a point on the number line such that the sum of its distance from 3, -1 and 0 is 10. Let's say, if $x = 0$,



Distance of 0 from 3 = 3

Distance of 0 from -1 = 1

Distance of 0 from 0 = 0

Sum of distance of 0 from 3, -1 and 0 is $3 + 1 + 0 = 4$.

So we know that x is not equal to 0. Now let's see what happens if $x = 3$.

Sum of distances of 3 from 3, -1 and 0 is $0 + 4 + 3 = 7$. We need 3 more units of distance to make it 10. We need to make x go a little more to the right. Tell me, what happens when x goes 1 unit to the right? By how much will the distance increase? By 3 units! Because x will be 1 unit away from each of the 3 points. When $x = 4$, sum of distances of 4 from 3, -1 and 0 is $1 + 5 + 4 = 10$.

So 4 is definitely one solution for x . What if we go further to the right? Every one unit further to right increases the

Quarter Wit_Quarter Wisdom- Part-1

distance by 3 units. So distance will keep increasing and will never be 10 again on this side of the number line.

Let's go to the other side. What happens if $x = -1$? Sum of distances of -1 from 3, -1 and 0 is $4 + 0 + 1 = 5$. To increase the distance, we need to go further to left. Remember, the same logic holds here – Every one step to left will increase the distance by 3 units. We need to increase the distance by 5 units. So we take 1 step to the left (reach -2) and then take $2/3^{\text{rd}}$ of a step to the left (reach -2.667). So $x = -2.667$ is another solution. Now, every time we take another step to the left, the total distance will increase.

Therefore, there are only two solutions for x : 4 and -2.667

To review:

$$|x - 3| + |x + 1| + |x| = 10$$



The Red line shows the region where the total distance of x from the 3 points is less than 10. The Blue lines show the region where the total distance of x from the 3 points is more than 10. The points -2.667 and 4 are the points where the total distance of x from the 3 points is 10.

Since there are 2 points where the total distance is 10, answer is (C).

A few things to ponder upon:

- I change the question to 'For how many values of x , is $|x - 3| + |x + 1| + |x| = 4$?' The answer now is 1. Why?
- I now change the question to 'For how many values of x , is $|x - 3| + 3|x + 1| + |x| = 4$?' The answer now is 0. Why?
- What happens if I change the question to 'For how many values of x , is $|2x - 3| + |x + 1| = 10$?'
- What if I change it to 'For how many values of x , is $|x - 3| - |x + 1| + |3x| = 10$?'

Thoughts on the points above are welcome. By the way, if a doubt arises somewhere, feel free to let me know and I will get back to you.

10. A Simple Approach to Percentages

Today, I will take up a relatively simple topic – Percentages. It is extremely relevant for GMAT and your everyday life. For the critics amongst you, let me give an example: What does a 20% sale with an additional 25% off on the \$85 sweater that you have your eye on mean to you? Rather than flipping open your HP12C, blink your eyes and the answer will swim in front of you... Uhh... I mean, after I tell you what you have to do in that blink (There is always a catch!).

Let me begin by saying that a percentage is a fraction. A fraction where the denominator is always 100, but just a fraction nevertheless. 50% means 50 per 'cent' (cent being 100) or $50/100$ or 50 out of every 100.

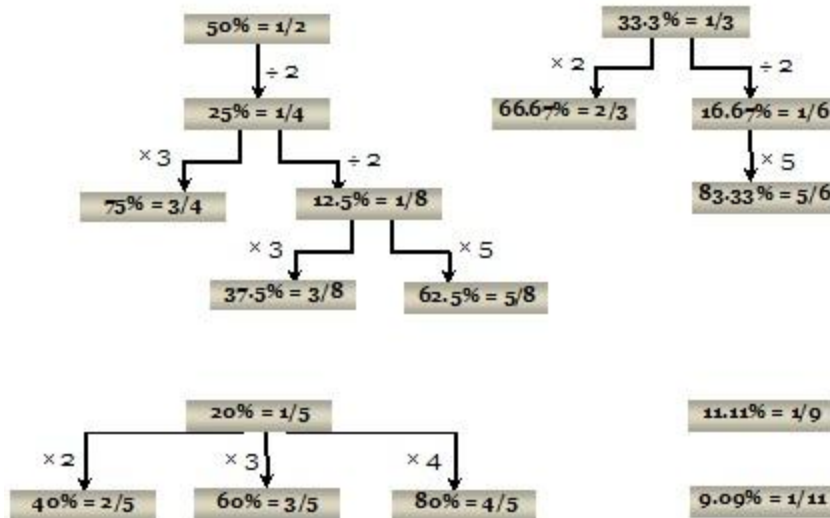
So, $50\% = 50/100 = \frac{1}{2}$

If I ask you what the 50% of 240 is, what will you do?

$(50/100)*240$ or $(1/2)*240$?

Quarter Wit_Quarter Wisdom- Part-1

I don't think I need to explain why the second representation is easier to handle. A fraction in its lowest form usually helps us to calculate faster. If this makes sense, think about 12.5% i.e. $12.5/100$. Let's say, I tell you that $12.5/100$ is equal to $1/8$. Now, if you had to find 12.5% of 64, what would be easier to do: $(12.5/100) * 64$ or $(1/8) * 64$? For this future ease, can you learn up some percent fraction equivalents today? In the interest of furtherance of this post, let's assume you answered with a resounding yes!



So now, if someone asks you, what is 60% of 350, all you should do is $(3/5) * 350 = 210$.

What if instead, the question is: If you increase 105 by 20%, what do you get?

You might be tempted to find the 20% of 105 and add it to 105. Something like this: $105 + (20/100) * 105 = 105 + 21 = 126$.

Or if you have been paying attention, then you might do it like this: $105 + (1/5) * 105 = 105 + 21 = 126$.

It is still not optimum! You did the calculation in two steps: In step 1, you found the $1/5^{\text{th}}$ of 105. In step 2, you added the $1/5^{\text{th}}$ to 105. Can we instead club it in a single step?

$$105 + (1/5) * 105 = 105 (1 + 1/5) = 105 * (6/5)$$

So when we have to increase a number by 20%, we just multiply it by $6/5$. When we have to decrease a number by 20%, we just multiply it by $4/5$ (because $1 - 1/5 = 4/5$). You might feel that these are little things, not likely to help you save much time. But once we build up on these little things, they work wonders.

Let's try to decrease 120 by 33.33%.

33.33% is $1/3$. When you try to decrease a number by $1/3$, you will need to multiply it by $(1 - 1/3) = 2/3$. So to decrease 120 by 33.33%, we just need to multiply it by $2/3$. We get $120 * (2/3) = 80$.

Try to do the following orally:

1. What is 40% of 320?
2. What do you get when you increase 352 by 37.5%?
3. What do you get when you decrease 819 by 11.11%?

The most important application of this method is successive percentage changes. Let's take an example:

Example: In 2008, the membership of People's Society was 90,000. In 2009, it increased by 22.22%. In 2010, it decreased by 9.09%. What was the membership at the end of 2010?

Quarter Wit_Quarter Wisdom- Part-1

Solution: $11.11\% = (1/9)$, therefore 22.22% must be $2/9$ (multiplying both sides of the equation by 2). To increase a number by 22.22% , we must multiply it by $11/9$ (because $1 + 2/9 = 11/9$). Also, $9.09\% = 1/11$. To decrease a number by 9.09% , we must multiply it by $10/11$ (because $1 - 1/11 = 10/11$)

Membership at the end of 2009 = $90,000 \times (11/9)$

Membership at the end of 2010 = $90,000 \times (11/9) \times (10/11) = 100,000$

The membership at the end of 2009 is the membership at the beginning of 2010 (i.e. $90,000 \times (11/9)$) which decreases by 9.09% .

Once you are comfortable with this method, you will directly jump to the step in Bold above.

And as for the \$85 sweater question, I will take it up in the next post on percentages.

11. Successive Percentage Changes.

Today, I will take a topic I briefly introduced in the [previous post on percentages](#). Let me start with the question I posted there.

What does a 20% sale with an additional 25% off on the \$85 sweater that you have your eye on mean to you?

It means a big rebate. Let's see how much:

If you reduce 85 by 20%, it becomes $85 \times 4/5 = \$68$. Now, you reduce it again by 25% and it becomes $68 \times (3/4) = \$51$

Notice that a 20% discount and then a 25% discount is not equal to a 45% discount ($85 \times 55/100 = \$46.75$). It is less than 45% but to the imperceptive, oblivious customer, it registers as 45% (Now you know why the retailers use the strategy of marking down by 20% and then giving an 'additional' 25% later!). The difference arises because the 20% discount was given on \$85 but the 25% discount was given on \$68. 25% of 68 is definitely less than 25% of 85 and hence the overall percentage decrease is less than 45%.

This is called successive percentage change – a number is changed by some percentage and then *the new number* is changed by another percentage. Both the percentage changes are not applied to the same original number.

The most popular example of successive percentage change is population change. Let us look at an example to understand this.

Example 1:

A city's population was 10,000 at the end of 2008. In 2009, it increased by 10% and in 2010, it decreased by 18.18%. What was the city's population at the end of 2010?

Solution:

Population at the end of 2008 = 10,000

Population at the end of 2009 = $10,000 \times (11/10) = 11,000$

Population at the end of 2010 = $11,000 \times (9/11) = 9000$

Simply put, population at the end of 2010 = $10,000 \times (11/10) \times (9/11) = 9000$

It is best to do the calculations in a single step because you do not need to calculate the intermediate population values. Besides, there is a good possibility that factors will get canceled out and hence, you will need to do fewer calculations.

Obviously, there is no limit to the number of successive percentage changes that can be made to a number. The approach remains unchanged in any case. Let me elaborate with another example:

Quarter Wit_Quarter Wisdom- Part-1

Example 2:

Six months back, the cost of an air ticket from Detroit to San Francisco was \$400. Four months back, the fares increased by 12.5%. Last month, the fares increased by 25% and yesterday, the airlines again increased the fares by 11.11%. What is the price of a Detroit to San Francisco ticket today?

Solution:

$$\text{Price of a ticket today} = 400 * (9/8) * (5/4) * (10/9) = \$625$$

This is much faster than finding the ticket price at every price change which would need the following steps:

$$\text{Price of a ticket four months back} = 400 * (9/8) = \$450$$

$$\text{Price of a ticket last month back} = 450 * (5/4) = \$562.5$$

$$\text{and finally, price of a ticket today} = 562.5 * (10/9) = \$625$$

So, in case you do not need the intermediate values, do not calculate them.

When there are only two successive percentage changes, we can derive a formula. In some cases, the formula makes the solution very simple.

When a number, N, changes by x% and then changes again by y%, we do the following to find the new number:

$$\text{New number} = N * (1 + x/100) * (1 + y/100)$$

$$\text{Now, } (1 + x/100) * (1 + y/100) = 1 + x/100 + y/100 + xy/10000$$

$$\text{If we say that } x + y + xy/100 = z, \text{ then } (1 + x/100) * (1 + y/100) = 1 + z/100$$

Here, **z is the effective percentage change** when a number is changed successively by two percentage changes. Let's take another example to see the formula in action:

Example 3:

A city's population was 10,000 at the end of 2008. In 2009, it increased by 20% and in 2010, it decreased by 10%. What was the city's population at the end of 2010?

$$x\% = 20\%$$

$$y\% = -10\% \text{ (Notice the negative sign here because this is a decrease)}$$

$$\text{Effective percentage change} = x + y + xy/100 = 20 + (-10) + 20*(-10)/100 = 8\%$$

$$\text{Population at the end of 2010} = 10,000 * (108/100) = 10800$$

Note: When the percentage is a decrease, a negative sign is used as shown above.

This formula is used only when there are two successive percentage changes and the percentages are easy to work with e.g. 15% and 25%, -10% and -30% etc.

With more than two successive percentage changes or trickier percentage values e.g. 11.11% and 18.18%, 9.09% and 6.25% etc, stick to the method shown above.

A major application of successive percentage changes in GMAT is the Markup-Discount-Profit questions. We will take that topic next week but I will leave you with a question to ponder upon:

Quarter Wit_Quarter Wisdom- Part-1

If a retailer marks up his goods by 40% and then offers a discount of 10%, what is his profit%?

12. Mark up, Discounts, & Profits :

Mark Up, Discount and Profit questions confuse a lot of people. But, actually, most of them are absolute sitters — very easy to solve — a free ride! How? We will just see. Let me begin with the previous post's question.

Question: If a retailer marks up an article by 40% and then offers a discount of 10%, what is his percentage profit?

Let us say the retailer buys the article for \$100 (\$100 is his cost price of the item). He marks it up by 40% i.e. increases his cost price by 40% ($100 * 140/100$) and puts a tag of \$140 on the article. Now, the article remains unsold and he puts it on sale – 10% off everything. So the article marked at \$140, gets \$14 off and sells at \$126 (because $140 * 9/10 = 126$). This \$126 is the selling price of the article. To re-cap, we obtained this selling price in the following way:

$$\text{Cost Price} * (1 + \text{Mark Up}\%) * (1 - \text{Discount}\%) = 100 * (1 + 40/100) * (1 - 10/100) = 126 = \text{Selling Price}$$

The profit made on the item is \$26 (obtained by subtracting 100, the retailer's cost price, from 126, the retailer's selling price).

He got a profit % of $(26/100) * 100 = 26\%$ (Profit/Cost Price x 100)

$$\text{Or we can say that } \text{Cost Price} * (1 + \text{Profit}\%) = 100 * (1 + 26/100) = 126 = \text{Selling Price}$$

The italicized parts above show the two ways in which you can reach the selling price: using mark-up and discount or using profit. The same thing is depicted in the diagram below:



$$\text{Therefore, } \text{Cost Price} * (1 + \text{Mark Up}\%) * (1 - \text{Discount}\%) = \text{Cost Price} * (1 + \text{Profit}\%)$$

Or

$$(1 + \text{Mark Up}\%) * (1 - \text{Discount}\%) = (1 + \text{Profit}\%)$$

Quarter Wit_Quarter Wisdom- Part-1

Look at the numbers here: Mark Up: 40%, Discount: 10%, Profit: 26% (Not 30% that we might expect because $40\% - 10\% = 30\%$)

Why? Because the discount offered was on \$140, not on \$100. So a bigger amount of \$14 was reduced from the price. Hence the profit decreased. This leads us to an extremely important question in percentages – What is the base? 100 was increased by 40% but the new number 140 was decreased by 10%. So in the two cases, the bases were different. Hence, you cannot simply subtract 10 from 40 and hope to get the Profit %. Also, mind you, almost certainly, 30% will be one of the answer choices, albeit incorrect. (The GMAT doesn't forego even the smallest opportunity of tricking you into making a mistake!)

Let's see this concept in action on a tricky third party question:

A dealer offers a cash discount of 20%. Further, a customer bargains and receives 20 articles for the price of 15 articles. The dealer still makes a profit of 20%. How much percent above the cost price were his articles marked?

- a) 100%
- b) 80%
- c) 75%
- d) $66\frac{2}{3}\%$
- e) 50%

This question involves two discounts:

1. the straight 20% off
2. discount offered by selling 20 articles for the price of 15 – a discount of cost price of 5 articles on 20 articles i.e. a discount of $\frac{5}{20} = 25\%$

Using the formula given above:

$$(1 + m/100)(1 - 20/100)(1 - 25/100) = (1 + 20/100)$$

$$m = 100$$

Therefore, the mark up was 100%.

Answer (A)

Note: The two discounts are successive percentage discounts.

Another application of successive percentage changes is the concept of compounding. But more on that, in the next post.

13.Simple Interest & The not so simple one

I am sure you have heard of the phenomenal “power of compounding.” Elders love to preach about the wisdom of starting a savings account at the age of 25 (when you don't have any money left from your pay check after your not-so-sensible

Quarter Wit_Quarter Wisdom- Part-1

Vuitton/Gucci/Chanel escapades!) rather than at the age of 40 (when you have 2 mortgages, 2 kids and a high maintenance Cadillac). Let's crunch some numbers to see if they are right.

Starting at age 25, if you put \$200 every month for 40 years at 10% per annum, you will have more than 1.25 million at the age of 65.

Starting at age 40, if you put \$200 every month for 25 years at 10% per annum, you will have only \$265,000 at the age of 65. You might have reservations about the fact that in the first case, you are investing more, after all! It's not just about the longer time period. So look at it another way.

If you invest \$25,000 at age 25 at 10% per annum, you will have \$1.13 million at age 65.

But if you invest \$25,000 at age 40 at 10% per annum, you will have \$271,000 at age 65.

In the first case, you invest for less than double the time (for 40 years as compared to 25 years in the second case) but you get four times the return (1.31 million as compared to 271,000)... It seems that elders might have been right about this after all. (Now I wish I had listened to my dad!)

Since compounding has a powerful influence on finances, it is something you will come across often during your MBA studies. So GMAT likes to test you on it too. Let's get crackin' then.

As I mentioned in my previous post, compound Interest is an important application of successive percentage changes.

GMAT tests you on simple and compound interest and sometimes, may even test you on the relation between the two. So let's look at both of them one by one.

When you say that the rate of simple interest is 10% per annum, it means that you earn 10% on your original principal every year. Say I deposit \$1000 for 4 years at 10% simple interest per annum. Amount at the end of one year is simply $1000 \times (11/10) = \$1100$ i.e. I earn \$100 in a year. Since it is simple interest, every year I will earn the same amount i.e. \$100. So total simple interest earned will be $\$100 \times 4 = \400 . If you observe carefully, we have calculated total simple interest using the following concept:

Simple Interest = Interest Rate/100 * Principal * No. of years

which is exactly what the formula for calculation of simple interest gives us.

Now let's go on to compound interest. Compounding means successive percentage changes. It means that a sum of money increases by a certain percentage in a year. At the end of the year, the interest earned is combined with the principal and next year, interest is earned on this combined amount.

Say I deposit \$1000 for 4 years at 10% compound interest per annum. Amount at the end of one year is simply $1000 \times (11/10) = \$1100$ i.e. I earn \$100 in a year. Till now it is just like simple interest. But from next year on, we will earn on this extra \$100 that we earned this year too. Amount at the end of 2nd year = $1100 \times (11/10) = 1210$. Amount at the end of third year will be $1210 \times (11/10)$ and so on... As you can see, this is just $1000 \times (11/10)(11/10)(11/10)(11/10)$ or

$$A = P(1 + r)^i$$

("i" is the number of time periods and r is the percentage rate of interest)

which is nothing but the 'amount in case of compound interest' formula

How does knowing this help us? GMAT tests your ingenuity and conceptual understanding. I will give below two examples where you will see how knowing this helps.

Example 1:

A bank launched a new financial instrument called VDeposit. A VDeposit offers you variable rate of compound interest in accordance with the current market rate. Ethan deposited \$8000 in a VDeposit. If he gets interest rates of 10% in the first two years and 12.5% in the third year, what is the total amount at the end of 3 years?

Quarter Wit_Quarter Wisdom- Part-1

As you can see, solving it using the standard formula is slightly cumbersome since we would have to use it twice. I would rather view it as:

$$8000 \times (11/10)(11/10)(9/8) = 1000 \times (11/10) \times (11/10) \times (9)$$

Notice that even though 12.5% compound interest was offered in the third year, we can still cancel off the 8 of 8000 with 8 of 12.5% increase when we view the calculation this way.

Amount = \$10,890

Example 2:

Mark deposited \$D in a scheme offering 5% simple interest per annum. Tetha deposited \$D in a scheme offering 5% compound interest per annum. At the end of second year, Tetha had earned a total of \$2.50 more than Mark. What is the value of D?

Till the end of first year, simple interest and compound interest cases are exactly the same. The difference comes in at the end of second year when compound interest offers interest on previous year's interest too. \$2.50 is 5% interest earned in the second year on first year's interest.

$$2.5 = (5/100) * I$$

$$I = \$50$$

So interest earned in the first year is \$50, which is 5% of the deposited amount D

$$50 = (5/100) * D$$

$$D = \$1000$$

In these and many more case, it pays to understand the concept of simple and compound interest.

Now, I will leave you with a question:

In the case of yearly compound interest, the ratio of amounts at the end of the 20th year to the amount at the end of the 22nd year is 0.81. What is the rate of interest?

14.The Sorcery of Ratios

Two trains, A and B, traveling towards each other on parallel tracks, start simultaneously from opposite ends of a 250 mile route. A takes a total of 3 hours to reach the opposite end while B takes a total of 2 hours to reach the opposite end. When train A meets train B during the journey, how far is train A from its starting point?

This question is not difficult but when I give it to my students, I see long, winding solutions, solutions that I find difficult to follow and after looking at a couple, my head starts to spin. Though, admittedly, most of them have the correct answer at the end. But you see, the correct answer alone is not enough. I like to see correct answers along with intuitive, intellectually stimulating methods. The most satisfying is seeing a blank rough sheet and the correct answer together. Rather than relying on the scratch pads and the markers, try and rely on your beautiful minds!

Now, if you haven't tried answering the above given question on your own yet, do it right away before reading the rest of the post.

So what kind of a solution do I like? The following:

Since A takes 3 hours to travel the same distance for which B takes 2 hours, time taken by A and B is in the ratio 3:2 so their speeds must be in the ratio 2:3. Hence, A will cover 2/5th of the total distance of 250 miles and B will cover the rest of the (3/5)th of the total distance. Therefore, A will be 100 miles from its starting point when A and B meet.

How much time do you think it takes one to think this through? 20-30 seconds at most! It just sounds so simple, doesn't it? But actually it involves a thorough understanding of quite a few fundamental concepts. If you take the time to get comfortable with these concepts, you will be able to save a lot of time on many Arithmetic questions. So let's delve into some basics.

Quarter Wit_Quarter Wisdom- Part-1

Ratios – One of the most basic and important concepts in the 'just figure it out' methodology.

A ratio gives a measure of the relation between two quantities. It doesn't give either of the two quantities. So score of A : score of B = 4:5 means that for every 4 points A scored, B scored 5 points. (It doesn't mean that A actually scored 4 and B actually scored 5.) To get the actual points scored by the two of them, we would need one of the following (or one of many other) scenarios:

Scenario 1:

Given: A scored 40 points. How many points did B score?

Obviously 50, right? If A scored 10 times 4, B must have scored 10 times 5, right?

Scenario 2:

Given: A and B together scored 90 points. How many points did A score and how many did B score?

If I think in ratio terms, A and B scored 4 and 5 respectively i.e. 9 in all. But actually, they scored 90 in all i.e. 10 times 9. So A must have scored 10 times 4 = 40 and B must have scored 10 times 5 = 50.

Scenario 3:

Given: B scored 10 points more than A. How many points did A score?

B:A is 5:4. For 4 points of A, B scored 1 extra. If B actually scored 10 points more than A, A must have scored 10 times 4 i.e. 40 points (and B must have scored 50 points)

In each of the cases above, you can think of 10 as the common multiplier. It helps you arrive at all the actual values. When you multiply the numbers in ratio terms with the multiplier, you get the actual values. Let us look at another example.

Weight of Adam : Weight of Sally = 7:5. Adam's weight is 133 pounds. What is Sally's weight?

The common multiplier is 19 (because $133/7 = 19$). Hence, Sally's weight is $5 \times 19 = 95$ pounds. The sum of their weights is $(7+5) \times 19 = 228$. The difference of their weights is $(7-5) \times 19 = 38$ pounds

Let us take an actual ratios question from our Arithmetic book now.

The ratio of Blue pens to Red pens is 5:7. When 3 Blue pens are added to the group and 9 Red pens are removed, the ratio of Blue pens to Red pens becomes 3:2. How many Red pens are there after the change?

- (A) 3
- (B) 5
- (C) 9
- (D) 12
- (E) 18

Let us say the multiplier of 5:7 is x.

So there must have been 5x Blue pens and 7x Red pens before the change.

After the change, number of Blue pens = $5x + 3$, number of Red pens = $7x - 9$

The ratio of these numbers is 3:2 which can also be represented as $3/2$.

$$(5x + 3)/(7x - 9) = 3/2$$

Quarter Wit_Quarter Wisdom- Part-1

Solving this, we get $x = 3$

After the change, there are $7 \times 3 - 9 = 12$ Red pens

This is the basic framework of ratios. Its uses lie in solving not only 'Ratio Questions' but also 'Weighted Average Questions', 'Distance Speed Time Questions', 'Work Rate Questions' etc. The benefit of using ratios is that you can get rid of the necessity of making and solving equations in many cases. For examples and explanations, check out subsequent posts (that is where I will explain the solution of the trains question given above too).

15.The Application of Ratios in TSD

Let's start with the applications of ratios today. Before you go on to an actual question, there is a relation between variables that you need to understand:

$$\text{Distance} = \text{Speed} \times \text{Time}$$

Now let's say my driving speed is a cool 100 mph. If I have to travel 100 miles, how much time will it take me? An hour, simple! Alright. If I have to travel 200 miles, how long will it take me? 2 hrs, you say? That is correct. What if I have to travel 500 miles? How long will it take me? 5 hrs, of course.. (now I am wasting your time, I know, but bear with me) When I hold my speed at a steady 100 mph, do you see a relation between Distance and Time? Can I say that if my distance doubles, my time taken doubles too? Can I say that if the distance that I have to travel on two different days is in the ratio 1: 5 (100 miles and 500 miles respectively), then the time I take on these two days will also be in the ratio 1:5 (1 hr and 5 hrs respectively)?

Yes, I can. Same is the relation of Distance with Speed keeping time constant. If this makes sense, you are half way there. We say Distance varies directly with Time (when Speed is kept constant) and Distance varies directly with Speed (when Time is kept constant). Anyway, we will take variation some other day.

Now, let me take another scenario.

Say, I have to travel 200 miles. If I keep a speed of 100 mph, I reach in 2 hrs. But what if the maximum speed allowed on the entire stretch is just 50 mph (and I am afraid of the law)? How long will it take me? Since my speed has reduced to half, it takes me double the time i.e. 4 hrs to cover 200 mph. So can I say that if there are two routes, both of 200 miles, and if my **speed on the routes is in the ratio 2:1** (100 mph and 50 mph respectively), then my **time taken will be in the ratio 1:2** (2 hrs and 4 hrs respectively)? We say that Speed varies inversely with Time (when Distance is kept constant). So if speed in two cases is in the ratio 3:7, time taken in the two cases will be in the ratio 7:3 (keeping the Distance same in both the cases)

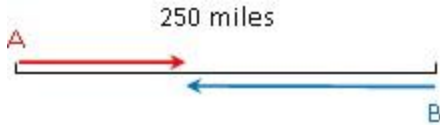
Now let's look at the question I posted in [the previous post](#):

Two trains, A and B, traveling toward each other on parallel tracks, started simultaneously from opposite ends of a 250 mile route. A takes a total of 3 hours to reach the opposite end while B takes a total of 2 hours to reach the opposite end. When train A meets train B during the journey, how far is train A from its starting point?



Both A and B travel a 250 mile route i.e. they travel the same distance. So the ratio of their speeds must be inverse of the ratio of time taken. (Say if train A's speed is 250 mph, it will take 1 hr and if train B's speed is 125 mph, it will take 2 hrs – ratio of speeds is 2:1, ratio of time taken is 1:2). Now we know that A takes 3 hrs to cover 250 miles and B takes 2 hrs to cover the same distance so ratio of time taken by A:B is 3:2. Then, ratio of their speeds must be 2:3. So for every 2 miles that A covers, B covers 3 miles in the same time.

Quarter Wit_Quarter Wisdom- Part-1



As shown in the [previous ratios post](#), the total distance in ratio terms will be 5 which is equal to 250 miles (this is the distance the trains have covered together when they meet). The multiplier is 50. So A covers a distance of $2 \times 50 = 100$ miles and B covers a distance of $3 \times 50 = 150$ miles. So train A is 100 miles away from its starting point! No equations needed.

Now, let's take another example.

The speed of bus A is 20% more than the speed of bus B. Bus B takes 2 hours longer than bus A to travel 600 miles. What is the speed of bus A?

Ordinarily, people would make equations and solve them to get to the answer. But we can do it quickly and orally.

Speed of bus A is 20% more than speed of bus B. This means that speed of bus A : speed of bus B is 120:100 i.e. 6:5. To travel the same distance, time taken by bus A: time taken by bus B will be 5:6. This difference of 1 in the ratio of time taken is actually given to be 2 hours. Hence, the multiplier is 2. Time taken by bus A to travel the 600 miles must be $5 \times 2 = 10$ hrs and time taken by bus B to travel the 600 miles must be $6 \times 2 = 12$ hrs.

Speed of bus A = $600/10 = 60$ mph

Try this logic on the next Time-Speed-Distance question you come across. Next week, we will look at the application of Ratios in Work problems.

16.Cracking the Work/ Rate Problems

Being comfortable with common ratios can save you a lot of time on the GMAT. [Last week](#) we covered distance/rate problems. Another great application of ratios is work rate problems. An important relation that helps us solve work rate problems is:

Work Done = Rate * Time

This relation will lead a perceptive observer to draw a parallel with another very popular relation most of us have come across:

Distance = Speed * Time

Speed is the same as Rate of work i.e. how fast you cover some distance or how fast you complete some given work. So obviously, if we can use ratios to solve many Distance Speed Time problems, we should be able to solve many Work Rate Time problems using ratios too.

Let's look at some examples. Try and solve each one of them on your own before you go through the solutions.

Example 1:

A tank has 5 inlet pipes. Three pipes are narrow and two are wide. Each of the three narrow pipes works at $\frac{1}{2}$ the rate of each of the wide pipes. All the pipes working together will take what fraction of time taken by the two wide pipes working together to fill the tank?

- (A) $\frac{1}{2}$
- (B) $\frac{2}{3}$
- (C) $\frac{3}{4}$
- (D) $\frac{3}{7}$
- (E) $\frac{4}{7}$

Quarter Wit_Quarter Wisdom- Part-1

We are given that rate of work of 1 narrow pipe : rate of work of 1 wide pipe = 1:2

If we can find the ratio of rate of work of 2 wide pipes : rate of work of all pipes together, then we can easily get the ratio of time taken by 2 wide pipes : time taken by all pipes together. This is because ratio of time taken will be inverse of the ratio of rate of work since work done in both the cases is the same. (For a further explanation of this concept, check out the previous post)

In ratio terms, rate of work of 3 narrow pipes is $1*3$ and rate of work of 2 wide pipes is $2*2$

Therefore, rate of work of 3 narrow pipes : rate of work of 2 wide pipes = 3:4

Or we can say rate of work of 2 wide pipes : rate of work of all pipes together = 4 : (3+4) = 4:7

Then, time taken by 2 wide pipes : time taken by all pipes together = 7:4 (i.e. inverse of 4:7)

So all the pipes together will take $\frac{4}{7}$ th of the time taken by the two wide pipes.

Answer (E)

Example 2:

Working at their respective constant rates, Paul, Abdul and Adam alone can finish a certain work in 3, 4, and 5 hours respectively. If all three work together to finish the work, what fraction of the work will be done by Adam?

- (A) $\frac{1}{4}$
- (B) $\frac{12}{47}$
- (C) $\frac{1}{3}$
- (D) $\frac{5}{12}$
- (E) $\frac{20}{47}$

It is given that:

Time taken by Paul : Time taken by Abdul : Time taken by Adam = 3:4:5

Rate of work must be inverse of time taken. But how do you take the inverse when you have a ratio of 3 quantities? Does it become 5:4:3? No. Actually it becomes $\frac{1}{3} : \frac{1}{4} : \frac{1}{5}$ (I will explain the 'why' for this when I take variation)

Rate of Paul : Rate of Abdul : Rate of Adam = $\frac{1}{3} : \frac{1}{4} : \frac{1}{5}$

Let's multiply this ratio by the LCM to convert it into integral form. The LCM of 3, 4 and 5 is 60.

Rate of Paul : Rate of Abdul : Rate of Adam = $(\frac{1}{3})*60 : (\frac{1}{4})*60 : (\frac{1}{5})*60 = 20:15:12$ (I would like to remind you here that multiplying or dividing each term of a ratio by the same number does not alter the ratio)

So if the total work is $20+15+12 = 47$ units, Adam will complete 12 units out of it. Hence the fraction of work done by Adam will be $\frac{12}{47}$.

Answer (B)

Example 3:

Machines A and B, working together, take t minutes to complete a particular work. Machine A, working alone, takes 64 minutes more than t to complete the same work. Machine B, working alone, takes 25 minutes more than t to complete the same work. What is the ratio of the time taken by machine A to the time taken by machine B to complete this work?

- (A) 5:8
- (B) 8:5

Quarter Wit_Quarter Wisdom- Part-1

- (C) 25:64
- (D) 25:39
- (E) 64:25

In this question, you can think logically to arrive at the answer quickly.

When machine A is working alone, it takes 64 extra minutes. Why? Because there is work leftover after t minutes. The work that would have been done by machine B in t minutes is leftover and is done by machine A in 64 minutes.

Time taken by A : Time taken by B = $64:t$ (I)

Similarly, when machine B works alone, it takes 25 extra minutes to complete the work that machine A would have done in t minutes.

Time taken by A : Time taken by B = $t:25$ (II)

From (I) and (II) above,

$$64/t = t/25$$

$$t = 40$$

Time taken by machine A : Time taken by machine B = $t:25 = 40:25 = 8:5$

Answer (B)

17.Heavily Weighted Weighted Averages

Today, I will delve into one of the most important topics (ubiquitous application) that are tested on GMAT. It is also one of the topics that will appear time and again during MBA e.g. in Corporate finance, you might be taught how to find '*Weighted Average Cost of Capital*'. So it will be highly beneficial if you have a feel for weighted average concepts.

The first question is – What is Weighted Average? Let me explain with an example.

A boy's age is 17 years and a girl's age is 20 years. What is their average age?

Simple enough, isn't it? Average age = $(17 + 20)/2 = 18.5$

It is the number that lies in the middle of 17 and 20. (Another method of arriving at this number would be to find the difference between them, 3, and divide it into 2 equal parts, 1.5 each. Now add 1.5 to the smaller number, 17, to get the average age of 18.5 years. Or subtract 1.5 from the greater number, 20, to get the average age of 18.5 years. But I digress. I will take averages later since it is just a special case of weighted averages.)

Now let me change the question a little.

There are 10 boys and 20 girls in a group. Average age of boys is 17 years and average age of girls is 20 years. What is the average age of the group?

Many people will be able to arrive at the following:

$$\text{Average Age} = (17*10 + 20*20)/(10 + 20) = 19 \text{ years}$$

Average age will be total number of years in the age of everyone in the group divided by total number of people in the group. Since the average age of boys is 17, so total number of years in the 10 boys' ages is $17*10$. Since the average age of girls is 20, the total number of years in the 20 girls' ages is $20*20$. The total number of boys and girls is $10 + 20$. Hence you use the expression given above to find the average age. I hope we are good up till now.

To establish a general formula, let me restate this question using variables and then we will just plug in the variables in place of the actual numbers above (Yes, it is opposite of what you would normally do when you have the formula and you plug in

Quarter Wit_Quarter Wisdom- Part-1

numbers. Our aim here is to deduce a generic formula from a specific example because the calculation above is intuitive to many of you but the formula is a little intimidating.)

There are w_1 boys and w_2 girls in a group. Average age of boys is A_1 years and average age of girls is A_2 years. What is the average age of the group?

$$\text{Average Age} = (A_1 \cdot w_1 + A_2 \cdot w_2) / (w_1 + w_2)$$

This is weighted average. Here we are not finding the average age of 1 boy and 1 girl. Instead we are finding the average age of 10 boys and 20 girls so their average age will not be 18.5 years. Boys have been given less weightage in the calculation of average because there are only 10 boys as compared to 20 girls. So the average has been found after accounting for the weightage (or 'importance' in regular English) given to boys and girls depending on how many boys and how many girls there are. Notice that the weighted average is 19 years which is closer to the average age of girls than to the average age of boys. This is because there are more girls so they 'pull' the average towards their own age i.e. 20 years.

Now that you know what weighted average is and also that you always knew the weighted average formula intuitively, let's move on to making things easier for you (Tougher, you say? Actually, once people know the scale method that I am going to discuss right now (It has been discussed in our Statistics and Problem Solving book too), they just love it!)

$$\text{So, Average Age, } A_{\text{avg}} = (A_1 \cdot w_1 + A_2 \cdot w_2) / (w_1 + w_2)$$

$$\text{Now if we re-arrange this formula, we get, } w_1/w_2 = (A_2 - A_{\text{avg}}) / (A_{\text{avg}} - A_1)$$

So we have got the ratio of weights w_1 and w_2 (the number of boys and the number of girls). How does it help us? Knowing this ratio, we can directly get the answer. Another example will make this clear.

John pays 30% tax and Ingrid pays 40% tax. Their combined tax rate is 37%. If John's gross salary is \$54,000, what is Ingrid's gross salary?

Here, we have the tax rate of John and Ingrid and their average tax rate. $A_1 = 30\%$, $A_2 = 40\%$ and $A_{\text{avg}} = 37\%$. The weights are their gross salaries – \$54,000 for John and w_2 for Ingrid. From here on, there are two ways to find the answer. Either plug in the values in the formula above or use the scale method. We will take a look at both.

1. Plug in the formula

$$w_1/w_2 = (A_2 - A_{\text{avg}}) / (A_{\text{avg}} - A_1) = (40 - 37) / (37 - 30) = 3/7$$

Since A_1 is John's tax rate and A_2 is Ingrid's tax rate, w_1 is John's salary and w_2 is Ingrid's salary

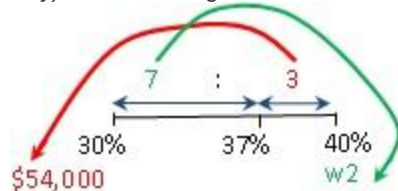
$$w_1/w_2 = \text{John's Salary} / \text{Ingrid's Salary} = 3/7 = 54,000 / \text{Ingrid's Salary}$$

$$\text{So Ingrid's Salary} = \$126,000$$

It should be obvious that either John or Ingrid could be A_1 (and the other would be A_2). For ease, it's a good idea to denote the larger number as A_2 and the smaller as A_1 (even if you do the other way around, you will still get the same answer)

2. Scale Method

On the number line, put the smaller number on the left side and the greater number on the right side (since it is intuitive that way). Put the average in the middle.



The distance between 30 and 37 is 7 and the distance between 37 and 40 is 3 so $w_1:w_2 = 3:7$ (As seen by the formula, the ratio is flipped).

$$\text{Since } w_1 = 54,000, w_2 \text{ will be } 126,000$$

$$\text{So Ingrid's salary is } \$126,000.$$

This method is especially useful when you have the average and need to find the ratio of weights. Check out next week's post for some 700 level examples of weighted average.

18.Zap the Weighted Average Brutes

Let me start today's discussion with a question from our Arithmetic book. I love this question because it is very crafty (much like actual GMAT questions, I assure you!) It looks like a calculation intensive question and makes you spend 3-4 minutes (scribbling furiously) but is actually pretty straight forward when understood from the 'weighted average' perspective. We looked at an easier version of this question in [the last post](#).

John and Ingrid pay 30% and 40% tax annually, respectively. If John makes \$56000 and Ingrid makes \$72000, what is their combined tax rate?

- a. 32%
- b. 34.4%
- c. 35%
- d. 35.6%
- e. 36.4%

If we do not use weighted averages concept, this question would involve a tricky calculation. Something on the lines of:

$$\text{Total Tax} = (30/100)*56000 + (40/100)*72000$$

$$\text{Tax Rate} = \text{Total Tax} / (56000 + 72000)$$

But we know better! The big numbers – 56000 and 72000 are just a smokescreen. I could have as well given you \$86380 and \$111060 as their salaries; I would have still obtained the same average tax rate! What is important is not the actual values of the salaries but the relation between the values i.e. the ratio of their salaries. Let me show you.

We need to find their average tax rate. Since their salaries are different, the average tax rate is not $(30 + 40)/2$. We need to find the 'weighted average of their tax rates'. In [the last post](#), we discussed

$$w1/w2 = (A2 - Aavg) / (Aavg - A1)$$

$$\text{The ratio of their salaries } w1/w2 = 56000 / 72000 = 7/9$$

$$7/9 = (40 - Tavg) / (Tavg - 30)$$

$$Tavg = 35.6\%$$

Imagine that! No long calculations! In the last post, when we wanted to find the average age of boys and girls – 10 boys with an average age of 17 yrs and 20 girls with an average age of 20 yrs, all we needed was the relative weights (relative number of people) in the two groups i.e. 1:2. It didn't matter whether there were 10 boys and 20 girls or 100 boys and 200 girls. It's exactly the same concept here. It doesn't matter what the actual salaries are. We just need to find the ratio of the salaries. Also notice that the two tax rates are 30% and 40%. The average tax rate is 35.6% i.e. closer to 40% than to 30%. Doesn't it make sense? Since the salary of Ingrid is \$72,000, that is, more than salary of John, her tax rate of 40% 'pulls' the average toward itself. In other words, Ingrid's tax rate has more 'weight' than John's. Hence the average shifts from 35% to 35.6% i.e. toward Ingrid's tax rate of 40%.

Let's now look at PS question no. 148 from the Official Guide which is a beautiful example of the use of weighted averages.

*If a, b and c are positive numbers such that $[a/(a+b)]*20 + [b/(a+b)]*40 = c$ and if $a < b$, which of the following could be the value of c?*

- (A) 20
- (B) 24
- (C) 30
- (D) 36
- (E) 40

Let me tell you, it isn't an easy question (and the explanation given in the OG makes my head spin).

First of all, notice that the question says: 'could be the value of c' not 'is the value of c' which means there isn't a unique value of c. 'c' could take multiple values and one of those is given in the options. Secondly, we are given that $a < b$. Now how does

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that figure in our scheme of things? It is not an equation so we certainly cannot use it to solve for c. If you look closely, you will notice that the given equation is

$$(20*a + 40*b) / (a + b) = c$$

Does it remind you of something? It should, considering that we are doing weighted averages right now! Isn't it very similar to the weighted average formula we saw in the last post?

$$(A1*w1 + A2*w2) / (w1 + w2) = \text{Weighted Average}$$

So basically, c is just the weighted average of 20 and 40 with a and b as weights. Since $a < b$, weightage given to 20 is less than the weightage given to 40 which implies that the average will be pulled closer to 40 than to 20. So the average will most certainly be greater than 30, which is right in the middle of 40 and 20, but will be less than 40. There is only one such number, 36, in the options. 'c' can take the value '36' and hence, (D) will be the answer. Elementary, isn't it? Not really! If you do not consider it from the weighted average perspective, this question can torture you for hours.

These are just a couple of many applications of weighted average. Next week, we will review Mixtures, another topic in which weighted averages are a lifesaver!

19. Don't Get mixed-up in Mixtures

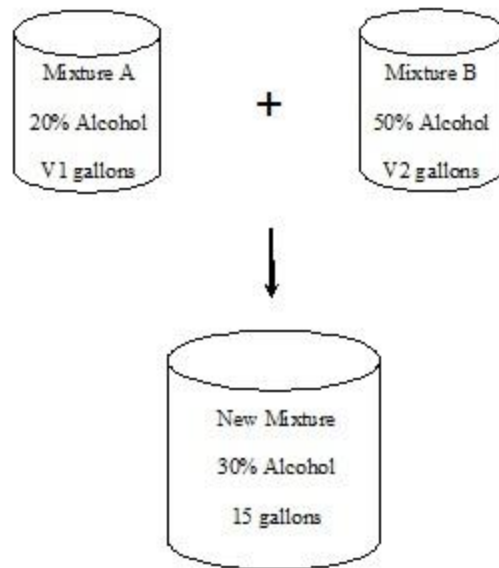
I visit the GMAT Club forum regularly and discuss some ideas, some methodologies there. The weighted averages method I discussed in my previous two posts is one of my most highly appreciated inputs on the forum. People love how easily they can solve some of the most difficult questions by just drawing a scale or using a ratio. If you are not a Quant jock, I am sure you feel a chill run down your spine every time you see a mixtures problem. But guess what, they are really simple if you just use the same weighted average concepts we discussed in the previous two posts. Let's look at a mixtures question in detail:

Mixture A is 20 percent alcohol, and mixture B is 50 percent alcohol. If the two are poured together to create a 15 gallon mixture that contains 30 percent alcohol, approximately how many gallons of mixture A are in the mixture?

- (A) 3 gallons
- (B) 4 gallons
- (C) 5 gallons
- (D) 10 gallons
- (E) 12 gallons

Let's try and imagine what is taking place here:

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Imagine two tumblers, one containing Mixture A and the other containing Mixture B. The mixtures from the two tumblers are poured into a new big empty tumbler. The first thing that we can conclude from this step is that $V1 + V2$ gallons = 15 gallons because the total volume when you combined the mixtures turned out to be 15 gallons (in the big tumbler).

Can you say anything about the relative value of $V1$ and $V2$? Can you say which one will be greater? We discussed in the previous post that when we find the weighted average of two quantities, the average is closer to the one which has higher weight because that quantity 'pulls' the average toward itself. So if the new mixture contains 30% alcohol, which of the two mixtures – A (20% alcohol) or B (50% alcohol), will have higher volume? Obviously A because it pulled the average toward itself. The average, 30% alcohol, is closer to 20% than to 50%. Now, let's calculate exactly how much of the 15 gallons was mixture A and how much was mixture B.

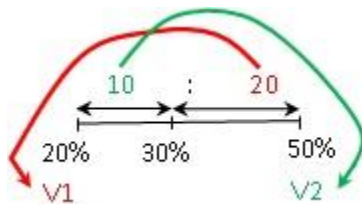
$$w1/w2 = (A2 - Aavg)/(Aavg - A1)$$

$$w1/w2 = (50 - 30)/(30 - 20) = 2/1$$

Ratio of volume of mixture A : volume of mixture B = 2:1.

So out of a total of 15 gallons, 10 gallons will be mixture A and 5 gallons will be mixture B. (Check out the post on [Ratios](#) if you are not sure how we arrived at this.)

Let's try the same thing using the scale method:



$$V1 : V2 = 2:1$$

So out of a total of 15 gallons, 10 gallons will be mixture A and 5 gallons will be mixture B. It wasn't very bad, was it?

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And now, let's try a quick example that looks ominous. People end up making tons of equations with 'x' and 'y' to solve this one but we can do it in a moment!

There are 2 bars of copper-nickel alloy. One bar has 2 parts of copper to 5 parts of nickel. The other has 3 parts of copper to 5 parts of nickel. If both bars are melted together to get a 20 kg bar with the final copper to nickel ratio of 5:11. What was the weight of the first bar?

- (A) 1 kg
- (B) 4 kg
- (C) 6 kg
- (D) 14 kg
- (E) 16 kg

This question might look different from the question above but it is actually very similar to it. Focus on any one of the two elements, say Copper (or Nickel if you fancy it more. Either way, you will get the same answer). Forget about the other one. If you notice, in the question above there are two elements in each mixture too (obviously, that is why they are called mixtures!) – Alcohol and Water. 20% Alcohol means that the rest 80% is Water. We just work with the concentration of Alcohol because it is given explicitly to us. We could as well say that mixture A has 80% water and mixture B has 50% water and the combined mixture has 70% water. Still the ratio obtained will be 2:1. But, still, somehow, giving the ratio of both Copper and Nickel in this question throws people off.

Let me quote one of my favorite GMAT instructors here: “And this is one of many reasons why McDonald's, the greatest marketing organization on earth, is a lot like the GMAT, the test that can help you get into the greatest marketing MBA programs on earth. They know how to make those tiny tweaks that make massive differences, at least in perception to the end consumer.” (Check out the complete(-ly) delicious article that discusses Shamrock Shakes [here](#).)

Anyway, let's get back to this question. I am keen to show you that it is, in fact, just like the previous question.

First bar is $2/7^{\text{th}}$ copper (2 parts copper and 5 parts nickel to get a total of 7 parts). Second bar is $3/8^{\text{th}}$ copper and the combined alloy is $5/16^{\text{th}}$ copper.

Now, let's calculate the ratio of weights of the two bars.

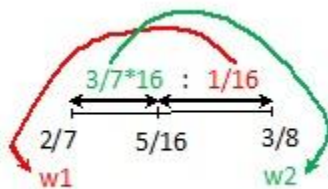
$$w1/w2 = (A2 - Aavg)/(Aavg - A1)$$

$$w1/w2 = (3/8 - 5/16)/(5/16 - 2/7) = 7/3$$

So weight of first bar: weight of second bar = 7:3

Out of a total of 20 kgs, the weight of the first bar was 14 kg and the weight of the second bar was 6 kg.

Or we can use the scale method too.



So the ratio of their weights is $1:3/7$ or 7:3

Out of a total of 20 kgs, the weight of the first bar was 14 kg and the weight of the second bar was 6 kg.

Hope these questions don't look very intimidating now. Use this method for your mixture problems; they won't feel like problems any more!

20.Divisibility Unraveled

Today, I will start the topic of Divisibility. We will discuss what divisibility is at a very basic level this week and then move on to remainders in the coming weeks.

So the first question is — what is division? Don't tell me what to do to divide a number by another, tell me why you do it. What is it that you are achieving by dividing one number by another? Let me tell you what I think — I like to think that division is grouping. Not happy? Let's look at an example then.

When I divide 6 by 2, I am actually just making groups of 2 from 6 and subtracting them.

$6 - 2 = 4$ (One group of 2 subtracted.)

$4 - 2 = 2$ (Another group of 2 subtracted.)

$2 - 2 = 0$ (Third group of 2 subtracted.)

I subtracted three groups so I get 3 as my quotient. Nothing is leftover so remainder is 0.

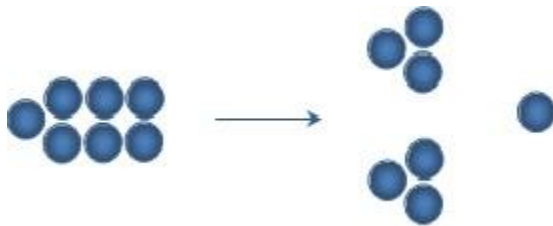
If I were to divide 14 by 3, I would subtract 4 groups of 3 from 14 giving me a quotient of 4, and 2 will be leftover. Hence, remainder will be 2. Hopefully, this made sense to you. Now let's look at division from the 'divisibility angle'.

Is 82 divisible by 4?

What I mean to ask here is that can I subtract groups of 4 from 82 such that nothing is leftover? When I subtract 20 groups of 4 (=80) from 82, I am left with 2. I cannot make any more groups of 4. Since we have a remainder of 2, 82 is not divisible by 4.

I have a fascination for diagrams. It stems in part from the fact that I wanted to be an artist but found that I have no talent in the field, and in part from the fact that I believe that diagrams help people understand the concepts better. On that note, let me draw a few diagrams to help you understand divisibility. Thereafter, you will be able to solve the tricky remainder questions quickly.

Imagine numbers as balls. When I say 7, imagine 7 balls in a group. Let's start easy: Is 7 divisible by 3? Before answering, think of the following figure:



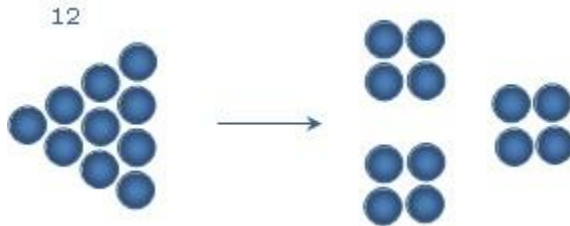
To divide by 3, I have to split 7 into groups of 3. I can make two groups of 3 and then 1 ball is leftover. Hence, when I divide 7 by 3, quotient is 2 and remainder, the ball that could not be put in a group, is 1.

Similarly, what happens when I divide 12 by 3?

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I get 4 groups and nothing is leftover. (Is this then the diagrammatic representation of $12 = 3 \times 4$? Sure.) Then when I divide 12 by 4, I should be able to get 3 groups with nothing leftover.

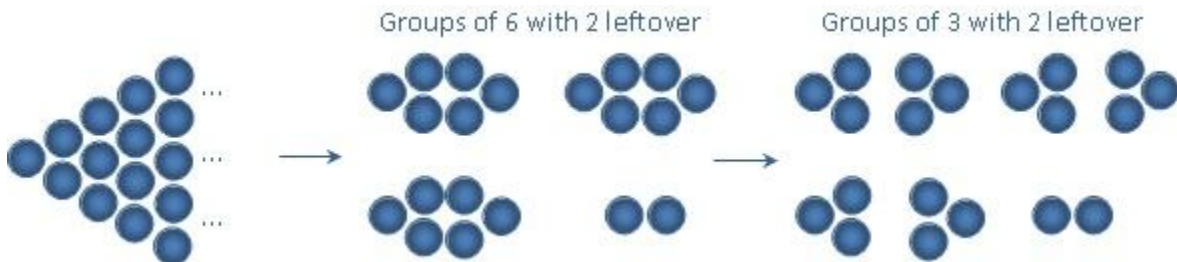


I guess you get the picture!

Now, a question: I have a number that when divided by 6, leaves a remainder of 2. What will be the remainder when the number is divided by 3?

(Do you think I moved on too quickly to too difficult a concept? Not really, just follow me)

So here, I do not know what the number is, but I know that when I make groups of 6 balls each, 2 balls are leftover (as shown in the diagram below). Logically, it follows that when I split each group of 6 balls into two groups of 3 balls, I will still have the same 2 balls remaining.



Since there are 2 balls leftover when groups of 3 are made, we can say that when we divide this number by 3, the remainder will still be 2! You don't have to make the diagram in the exam of course. Just imagine it.

A slight twist on the question above: I have a number that when divided by 6, leaves a remainder of 4. What will be the remainder when the number is divided by 3? Imagine the picture. Just like above but with groups of 6, there are 4 balls leftover. You divide the groups of 6 into groups of 3 – all is well till now – but then, you make another group of 3 from the 4 balls that are leftover. Therefore, only one ball will be leftover. Hence, when you divide the given number by 3, the remainder will be 1!

Try a few more examples and let this theory sink in. We will delve into more details in the coming few weeks.

21.Divisibility tested in GMAT

Let's continue from where we left the [last post on divisibility problems on the GMAT](#). I will add another level of complexity to the last question we tackled in that post.

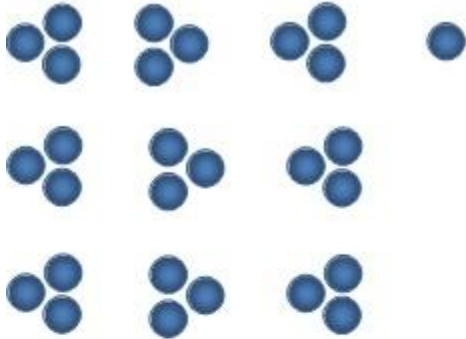
Question: A number when divided by 3 gives a remainder of 1. How many distinct values can the remainder take when

Quarter Wit_Quarter Wisdom- Part-1

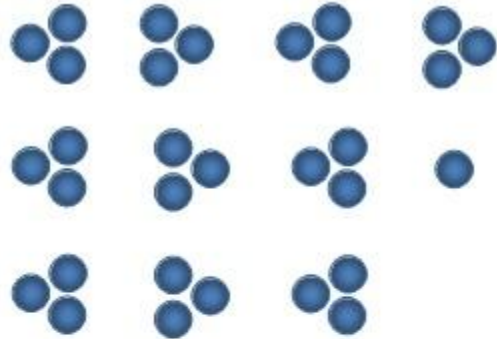
the same number is divided by 9?

Now imagine that there are lots of groups of 3 and 1 ball is leftover. We don't know exactly how many groups of 3 there are. There could be zero and there could be a 100 but let's assume that there are many. It would look something like this:

Groups of 3 with 1 leftover



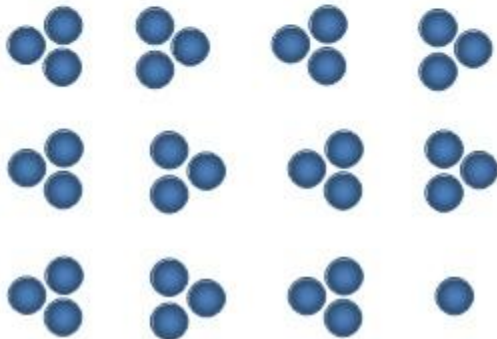
Groups of 3 with 1 leftover



Or

Or

Groups of 3 with 1 leftover



etc...

Now, we have to make groups of 9 out of these so we start combining three groups of 3s to make a group of 9. Let's see what the possibilities at the end are.

1. All groups of 3s get used to make groups of 9 and the 1 ball from before is again leftover.

Groups of 9 with 1 leftover



2. One group of 3 and the 1 ball from before, giving you a total of 4 balls, are leftover.

Quarter Wit_Quarter Wisdom- Part-1

Groups of 9 with 4 leftover



3. Two groups of 3 and the 1 ball from before, giving you a total of 7 balls, are leftover.

Groups of 9 with 7 leftover



(Three or more groups of 3 cannot be leftover because then, we will be able to make another group of 9 out of them. This is the reason why the remainder will never be 9 or greater than 9.)

Therefore, you can have the remainder in three distinct ways: 1, 4 and 7.

Now, let's apply what we have learned to a GMAT Data Sufficiency question.

Question: What is the remainder when n is divided by 26, given that n divided by 13 gives "a" as the quotient and "b" as the remainder? (a, b and n are positive integers)

- (1) a is odd
- (2) $b = 3$

This means that out of " n " balls, if we make groups of 13, we will be able to make "a" groups and will have "b" balls leftover.

What happens when we try to combine two groups of 13 to make a group of 26? There are two possibilities: all groups of 13 will be used to make groups of 26 and "b" balls will be leftover (as before) or one group of 13 and "b" balls will be leftover.

What will decide whether a group of 13 will be leftover? If "a" is 2 i.e. we have two groups of 13, we will be able to make one group of 26 and no group of 13 will be leftover. If "a" is 3 i.e. we have three groups of 13, we will be able to make one group of 26 and one group of 13 will be leftover. If "a" is 4 i.e. we have four groups of 13, we will be able to make two groups of 26 and no group of 13 will be leftover. What do you conclude from these examples? If "a" is even, we will have no group of 13 leftover. If "a" is odd, we will have one group of 13 leftover. So the remainder when n is divided by 26 will depend on whether a is odd or even and the value of "b." Let us look at the statements now:

Statement 1: a is odd.

If "a" is odd, then a group of 13 will be leftover. So the remainder will be $13 + b$. But we do not know the value of "b." So this statement alone is not sufficient.

Quarter Wit_Quarter Wisdom- Part-1

Statement 2: $b = 3$

We now know that $b = 3$ but from this statement alone, we do not know whether a is odd or even. So this statement alone is not sufficient.

Taking both statements together, we know that remainder is $13+3 = 16$. Hence both statements together are sufficient.

Answer (C).

I do hope that this concept is quite clear to you now. We will look at some other remainder concepts in future blog posts!

22.Divisibility applied to Remainders

Let's continue our discussion of Divisibility and Remainders. If you have been preparing for GMAT for a while, I am sure you would have come across a question of the following form:

Question: When positive integer n is divided by 3, the remainder is 1. When n is divided by 7, the remainder is 5. What is the smallest positive integer p , such that $(n + p)$ is a multiple of 21?

- (A) 1
- (B) 2
- (C) 5
- (D) 19
- (E) 20

Let us try and understand what the question is saying using what we have learnt so far.

"When positive integer n is divided by 3, the remainder is 1."

When I read this, the following is what comes to my mind:



"When n is divided by 7, the remainder is 5."

This statement brings this image to mind:

Quarter Wit_Quarter Wisdom- Part-1

n makes groups of 7 with 5 leftover



Now, before we move ahead, we need to digress a little.

Let us say, we have a number N which is divisible by 3 and by 7. Can we say it will be divisible by 21, the LCM of 3 and 7? Sure! Since the number is divisible by both 3 and 7, it has factors of 3 and 7 in it i.e. it has 21 as a factor. Let us try to analyze this situation from the 'diagram perspective' we have learned recently too. When we divide N by 3, the quotient will be divisible by 7. Since the quotient decides how many groups we get, when we make groups of 3, we will get 7 groups or 14 groups or 21 groups etc. Hence we will be able to club each of the 7 groups to make groups of 21. Hence, N will be completely divisible by 21.

Let's consider another scenario now.

If we have a number N, which when divided by 3 gives a remainder 1 and when divided by 7 gives a remainder 1, what would be the remainder when N is divided by 21?

N would be of the form:

$$N = 3a + 1 \text{ (groups of 3 with 1 leftover) and}$$

$$N = 7b + 1 \text{ (groups of 7 with 1 leftover)}$$

N would be one of the following numbers: 3 + 1, 6 + 1, 9 + 1, 12 + 1, 15 + 1, 18 + 1, **21 + 1** etc

It would also be found in this list: 7 + 1, 14 + 1, **21 + 1**, 28 + 1, 35 + 1 etc

So when N is divided by the LCM, 21, it will give 1 as remainder (as is apparent above). What we conclude from this is that if N gives the same remainder with two numbers, it will give the same remainder for their LCM too. Why? Try to use the diagrams perspective now. If we make groups of 3, we will get 7 groups or 14 groups etc and 1 will be leftover. If we club each of the 7 groups of 3 together to make groups of 21, we will still have 1 leftover. Hence when N is divided by 21, the remainder will still be 1.

Now, let's come back to our original question. We have a number n which when divided by 3 gives a remainder 1 and when divided by 7 gives a remainder 5. We can say the number is of the form:

$$n = 3a + 1$$

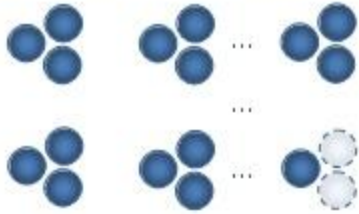
and

$$n = 7b + 5$$

Let me show you some diagrams here since the concept involved is a little tricky:

Quarter Wit_Quarter Wisdom- Part-1

n makes groups of 3 with 1 leftover.
We need 2 to make another group.



n makes groups of 7 with 5 leftover.
We need 2 to make another group.



Can we say that the remainder in both the cases is (-2) since we need another 2 to make complete groups of 3 and 7? When n is divided by 3 and the remainder obtained is 1, it is the same as saying the remainder is -2. n is 1 more than a multiple of 3 which means it is 2 less than the next multiple of 3. Therefore, we can say $n = 3x - 2$ and $n = 7y - 2$.

Now this is exactly like the situation we discussed above. When we divide n by 21, remainder will be -2, i.e. the same remainder. Does it mean that if we clubbed 7 groups of 3 together to make groups of 21, we would need 2 more to complete the last group of 21? I hope you will say 'yes'. Does this also mean that we need 2 more to make n divisible by 21? Yes, of course. Hence p must be 2.

Next week, we will discuss a trickier version of this question which needs some additional concepts. Till then, practice some similar questions to gain confidence in this concept.

23.Divisibility applied to Remainders-II

Let's continue on our endeavor to understand divisibility and remainders in this post. [Last week's post](#) focused on situations where the remainders were equal. Today, let's see how to deal with situations where the remainders are different.

Question: When positive integer n is divided by 3, the remainder is 2. When n is divided by 7, the remainder is 5. How many values less than 100 can n take?

- (A) 0
- (B) 2
- (C) 3
- (D) 4
- (E) 5

So n is a number 2 greater than a multiple of 3 (or we can say, it is 1 less than the next multiple of 3). It is also 5 greater than a multiple of 7 (or we can say it is 2 less than the next multiple of 7)

$$n = 3a + 2 = 3x - 1$$

$$n = 7b + 5 = 7y - 2$$

No common remainder! When we have a common remainder, the smallest value of n would be the common remainder. Say, if n were of the forms: $(3a + 1)$ and $(7b + 1)$, the smallest number of both these forms is 1. When 1 is divided by 3, the quotient is 0 and the remainder is 1. When 1 is divided by 7, the quotient is 0 and the remainder is 1. But that is not the case here. So then, what do we do now? Let's try and work with some trial and error now. n belongs to both the lists given below:

Numbers of the form $(3a+2)$: 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50...

Numbers of the form $(7b + 5)$: 5, 12, 19, 26, 33, 40, 47, 54, 61, 68, 75...

Which numbers are common to both the lists? 5, 26, 47 and there should be more. Do you see some link between these numbers? Let me show you some connections:

- 26 is 21 more than 5.
- 47 is 21 more than 26.

Quarter Wit_Quarter Wisdom- Part-1

- 21 is the LCM of 3 and 7.

How do we explain these? Say, we identified that the smallest positive number which gives a remainder of 2 when divided by 3 and a remainder of 5 when divided by 7 is 5 (note here that when we divide 5 by 7, the quotient is 0 and the remainder is 5). What will be the next such number? Since the next number will also belong to both the lists above so it will be 3/6/9/12/15/18/21... away from 5 and it will also be 7/14/21/28/35/42... away from 5 i.e. it will be a multiple of 3 and a multiple of 7 away from 5. The smallest such multiple is obviously the LCM (lowest common multiple) of 3 and 7 i.e. 21. Hence the next such number will be 21 away from 5. We get 26. Use the same logic to get the next such number. It will be another 21 away from 26 so we get 47. By the same logic, the next few such numbers will be 68, 89, 110 etc. How many such numbers will be less than 100? 5, 26, 47, 68, 89 i.e. 5 such numbers.

So, does this mean that when the remainders are not equal, you will need to make the lists given above. Well yes, in a way, but you can do it mentally. Let me explain. In the first 100 numbers, there will be many more numbers of the form $(3a+2)$ than $(7b+5)$. Since n should be of both the forms, let's start checking in the smaller list first.

Say $b = 0$, the number is 5. Is 5 of the form $(3a+2)$. Yes! Your list has served its purpose. Now all we need to do is keep adding 21 (the LCM of 3 and 7) to get the next few numbers! This turned out to be an easy example. Let's change the values a little to make it a little more cumbersome.

Question: When positive integer n is divided by 13, the remainder is 2. When n is divided by 8, the remainder is 5. How many such values are less than 180?

- (A) 0
- (B) 1
- (C) 2
- (D) 3
- (E) 4

The LCM of 8 and 13 is 104. Hence there cannot be more than 2 such values less than 180. Options (D) and (E) are out of the window for sure.

The number n should be of the following two forms:

$$n = 8a + 5$$

$$n = 13b + 2$$

In a given bunch of numbers, there will be many more numbers of the form $(8a + 5)$ and fewer of the form $(13b + 2)$ so let's start with a number of the form $(13b + 2)$.

If $b = 0$, $n = 2$. Is it of the form $(8a + 5)$? No. $n/8$ gives a remainder of 2, not 5.

If $b = 1$, $n = 15$. Is it of the form $(8a + 5)$? No. $n/8$ gives a remainder of 7, not 5.

If $b = 2$, $n = 28$. Is it of the form $(8a + 5)$? No. $n/8$ gives a remainder of 4, not 5.

If $b = 3$, $n = 41$. Is it of the form $(8a + 5)$? No. $n/8$ gives a remainder of 1, not 5.

If $b = 4$, $n = 54$. Is it of the form $(8a + 5)$? No. $n/8$ gives a remainder of 6, not 5.

If $b = 5$, $n = 67$. Is it of the form $(8a + 5)$? No. $n/8$ gives a remainder of 3, not 5.

If $b = 6$, $n = 80$. Is it of the form $(8a + 5)$? No. $n/8$ gives a remainder of 0, not 5.

If $b = 7$, $n = 93$. Is it of the form $(8a + 5)$? Yes! $n/8$ gives a remainder of 5.

The smallest value of n is 93. The next value of $n = 93 + 104 = 197$ i.e. greater than 180. Hence there is just one value of n less than 180. Let me continue the steps above just for kicks...

If $b = 8$, $n = 106$. Is it of the form $(8a + 5)$? No. $n/8$ gives a remainder of 2, not 5.

If $b = 9$, $n = 119$. Is it of the form $(8a + 5)$? No. $n/8$ gives a remainder of 7, not 5.

Do you see that we have started getting the same remainders again in the same order: 2, 7...? 106 is 104 more than 2. 119 is 104 more than 15. Also notice that we got all possible remainders for 8 (0, 1, 2, 3, 4, 5, 6, 7) while n was less than 104, the LCM of 8 and 13. Can you reason it out? I will leave you here with your thoughts...

24.Knocking off the remaining remainders

For the past few weeks, we have been focusing on Divisibility and Remainders. There are some more 'types' of remainder questions. Let's take them one by one so that by the end of it all, you are an expert in everything related to remainders. In this post I will start with a question similar to what you might find in the *Official Guide for GMAT Review*.

Question: If a and b are positive integers such that $a/b = 97.16$, which of the following cannot be the remainder when a is divided by b ?

- (A) 4
- (B) 12
- (C) 22
- (D) 28
- (E) 96

Some of my most brilliant students have asked me something similar to this: "When I divide 11 by 4, I get 2.75. What do you mean by, 'What is the remainder?' Where is it?" So if it is bothering you as well, don't worry; I will address this issue first.

Say, I tell you the following: Divide 11 by 4. What do you get?

You could answer me with one of the following:

Case 1: You could say, "I get 2.75"

Case 2: You could say, "I get 2 as the quotient and 3 as the remainder."

Either ways, you are correct. $11/4 = (2 \frac{3}{4})$

When you use the decimal form, you get a .75 which you add to 2 to give you 2.75. This .75 is nothing but the way you express the remainder 3. When you divide 11 by 4, 4 goes into 11 two times and then 3 is left over. When 4 goes into 3, you get 0.75 which is $\frac{3}{4}$. That is the reason why you can write $11/4$ as $(2 \frac{3}{4})$ in mixed fractions.

Do the following calculations and see what you get. Express the answer in both the forms – Decimal form and quotient-remainder form.

1. Divide 22 by 8.
2. Divide 55 by 20.
3. Divide 275 by 100.

Let's see what we get in each case.

1. Divide 22 by 8.

Case 1: In decimal form we get 2.75

Case 2: In quotient-remainder form we get 2 as quotient and 6 as remainder (Remember, the divisor is 8 here). We can say $22/8 = 2 (6/8)$ in mixed fractions.

1. Divide 55 by 20.

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Case 1: In decimal form we get 2.75

Case 2: In quotient-remainder form we get 2 as quotient and 15 as remainder (Remember, the divisor is 20 here). We can say $55/20 = 2 \text{ (15/20)}$ in mixed fractions.

1. Divide 275 by 100.

Case 1: In decimal form we get 2.75

Case 2: In quotient-remainder form we get 2 as quotient and 75 as remainder (Remember, the divisor is 100 here). We can say $275/100 = 2 \text{ (75/100)}$ in mixed fractions.

Notice that the remainder is different in each case above. As the divisor changes, the remainder changes even though in decimal form the answer is the same. This is so because in each of the above cases, $6/8 = 0.75$, $15/20 = 0.75$ and $75/100 = 0.75$. Each of these fractions is equal to $\frac{3}{4}$. Now if I flip the question and say, "When I divide a by b, I get 2.75. Which of the following cannot be the remainder when a is divided by b: 2, 3, 6, 12, 15?"

We saw above that $2.75 = 2 \text{ (75/100)} = 2 \frac{3}{4}$ in the lowest form. So the remainder will be 3 if the divisor is 4. The remainder will be 6 if the divisor is 8. The remainder will be 15 if the divisor is 20. On the same lines, the remainder will be 12 if the divisor is 16 because $12/16 = \frac{3}{4}$. Can the remainder be 2?

$$3/4 = 2/?$$

We cannot have an integer value in place of the '?'. Hence we will never get 2 as the remainder. The remainder will always be a positive multiple of 3.

Let's go back to the original question now: If a and b are positive integers such that $a/b = 97.16$, which of the following cannot be the remainder when a is divided by b?

$$a/b = 97.16 = 97 \text{ (16/100)} \text{ in mixed fraction} = 97 \text{ (4/25)} \text{ in the lowest form.}$$

The remainder must be a positive multiple of 4. 22 is not a multiple of 4 hence you can never have 22 as the remainder

$$4/25 = 22/?$$

You cannot have an integer in place of the '?'

Hence answer is 22.

Let me end this post with a question for you: If a and b are positive integers such that $a/b = 82.024$, which of the following can be the value of b?

- (A) 100
- (B) 150
- (C) 200
- (D) 250
- (E) 550

Quarter Wit_Quarter Wisdom- Part-1

hopefully, you will arrive at the answer in a few seconds!

25.A remainders post for the geek in you

In this post, I would like to focus on a particular type of remainder questions and how to solve them in a particular way. For the type of questions I am going to discuss today, I like to use "Binomial Theorem." You might be tempted to run away right now and save yourself some precious time if you are not a Math geek but wait! We will just use an application of Binomial which I will explain in very simple language. I am quite certain that you will be comfortable with the method if you just give it a chance.

Question 1: What is the remainder when $(3^{84})/26$

- (A) 0
- (B) 1
- (C) 2
- (D) 24
- (E) 25

First up, GMAT questions don't involve any painful calculations. So my thought is that there has to be an obvious link between 3 and 26. 26 is 1 less than the cube of 3. (It helps one to know the squares of first 20 numbers and cubes of first 10 numbers.)

So, $3^3 = 27$

But how is it going to help us? Now we come to binomial theorem. Let me start with something you already know.

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3ba^2 + 3ab^2 + b^3$$

What about $(a + b)^4$ or $(a + b)^5$ or higher powers? Binomial theorem just tells us how to expand these expressions. It gives you a general formula:

$$(a + b)^n = a^n + n \cdot a^{(n-1)} \cdot b + \frac{n(n-1)}{2} \cdot a^{(n-2)} \cdot b^2 + \dots + n \cdot a \cdot b^{(n-1)} + b^n$$

I know the above looks intimidating but our concern is limited to the last term of the expression. Notice that every term above is divisible by 'a' except for the last term b^n . Every term but the last has 'a' as a factor. That is all you need to understand about Binomial Theorem.

Now for some quick applications:

What is the last term when you expand $(8 + 1)^{20}$? It is 1^{20} (which is just '1').

When you expand $(8 + 1)^{20}$, is every term divisible by 8? Yes, except for the last term, 1, because every term has 8 as a factor except for the last term.

If I divide $(8 + 1)^{20}$ by 8, what will be the remainder? Since every term (except for the last one) in the expansion of $(8 + 1)^{20}$ is divisible by 8, we can say that $(8 + 1)^{20}$ is 1 more than a multiple of 8. Hence the remainder when we divide it by 8 will be 1.

Or I can say that when I divide 9^{20} (which is just $(8 + 1)^{20}$) by 8, the remainder is 1.

Now let's look at our original question.

$$(3^{84}) = (3^3)^{28} = 27^{28} = (26 + 1)^{28}$$

Every term of $(26 + 1)^{28}$ will be divisible by 26 except for the last one. The last term will be $1^{28} = 1$. Hence, when you divide 27^{28} by 26, the remainder will be 1.

Answer (B).

All you had to do was to look for a power of 3 which is 1 more or 1 less than 26. We found that the third power of 3 is 1 more than 26. We adjusted the power to make 27 the base and split it into $(26 + 1)$. We got the remainder as 1. Why do we

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necessarily look for a power 1 more or 1 less? We do that because 1^n is always 1. If we are left with 2^{28} , we again have a problem since we don't know what 2^{28} is. Let's use this concept in another problem now:

Question 2: What is the remainder when 2^{86} is divided by 9?

- (A) 1
- (B) 2
- (C) 3
- (D) 4
- (E) 8

I have added a few complications in this question. Let's tackle them one by one. We start by looking for a power of 2 which is 1 more or 1 less than 9. We know $2^3 = 8$ which is 1 less than 9.

Next, let's adjust the power to make the base 8.

$$2^{86} = 8^?$$

86 is not divisible by 3. The closest integer less than 86 that is divisible by 3 is 84. So, separate out two 2s and work with the rest of the 84 2s as of now.

$$2^{86} = (2^2) * (2^{84}) = (4) * (2^3)^{28} = 4 * (8^{28})$$

I am going to forget about the 4 for the time being.

$$8^{28} = (9 - 1)^{28} = [9 + (-1)]^{28}$$

Every term of this expression will be divisible by 9 except for the last term $(-1)^{28}$ which is again equal to 1.

Hence, 8^{28} will give a remainder 1 when divided by 9.

I can say that $8^{28} = 9m + 1$ where m is some positive integer. Now, we need to consider the 4 that we left out in the previous step. Our actual expression is

$$4 * 8^{28} = 4 * (9m + 1) = 4*9m + 4$$

When I divide this by 9, $4*9m$ is divisible by 9. So, $4*9m + 4$ is 4 more than a multiple of 9. Hence the remainder will be 4.

A question to ponder on: How will you solve this question if I change it to "What is the remainder of 2^{83} is divided by 9?"

26.An intellectual exercise in TSD

Okay, let's move away from "Divisibility and Remainders" on the GMAT, at least temporarily, and let's focus our attention on another topic. If you have read some of my previous posts, I guess you know that I like to solve questions "logically." I like to avoid making equations. Instead, I try to make myself "figure it out."

A few days back, I came across a Time-Speed-Distance problem which was a perfect example of how you could "figure stuff out" without dealing with any equations. Actually, you can do that with a majority of GMAT questions (and save yourself loads of time!) but what was special about this question was that a couple of instructors of one of our competitor (I am not at liberty to disclose exactly who this competitor is!) had told their students that it is not possible to solve it logically. That got me thinking that perhaps, logical thinking is not as widely utilized as we at Veritas would like to believe. (I think our love for logical thinking is also apparent in the way we teach Sentence Correction!) Anyway, I thought of sharing the question and its logical solution with you! Here goes...

Now, before you look at the solution, try and solve it on your own. Do not use a pen/pencil/marker/any other writing instrument and don't try to make equations in your head. Just try and reason out the solution! It will be a great intellectual exercise and if you are able to get the answer by using your mind alone, let me know for a virtual pat on the back!

Two friends, Tanaya and Stephen were standing together. Tanaya begins to walk in a straight line away from Stephen at a constant rate of 3 miles per hour. One hour later, Stephen begins to run in a straight line in the exact opposite direction at a constant rate of 9 miles per hour. If both Tanaya and Stephen continue to travel, what is the positive difference between the amount of time it takes Stephen to cover the exact distance that Tanaya has covered and the amount of time it takes Stephen to cover twice the distance that Tanaya has covered?

Quarter Wit_Quarter Wisdom- Part-1

- (A) 60 mins
- (B) 72 mins
- (C) 90 mins
- (D) 100 mins
- (E) 120 mins

Solution:

Following is what goes on in my mind when I read the question. I will quote sentences from the question to explain you the thought process.

“Two friends, Tanaya and Stephen were standing together.”

When I read the above, I think, “Ok, so two people, Tanaya and Stephen, are standing at the same point at a particular time, say 12 noon.”

“Tanaya begins to walk in a straight line away from Stephen at a constant rate of 3 miles/hour.”

Now I think, “Stephen is still standing where they were standing together but Tanaya starts walking away from Stephen at a speed of 3 miles/hour. Women!”

“One hour later, Stephen begins to run in a straight line in the exact opposite direction at a constant rate of 9 miles per hour.”

Stephen was shell shocked! After a whole hour (i.e. at 1:00 pm), when Tanaya was actually 3 miles away from him, he regains control of his legs and starts running in the opposite direction at 9 miles/hr, three times the speed at which Tanaya was walking away (vengeance!).

“What is the positive difference between the amount of time it takes Stephen to cover the exact distance that Tanaya has covered and ...”

So, I need to find the time it takes Stephen to cover the exact distance that Tanaya has covered. To cover the same distance as Tanaya, Stephen needs to cover the distance that Tanaya is covering now plus he has to cover the extra 3 miles that Tanaya has already covered. The thing going for him is that he is running much faster than Tanaya. Out of his speed of 9 miles/hr, 3 miles/hr get used to cover what Tanaya is covering right now (since Tanaya's speed is 3 miles/hr). So he uses the extra 6 miles/hr to cover the 3 miles that Tanaya has already covered! That means it will take him half an hour (3 miles/6 miles/hr) to cover the extra distance. In half an hour, i.e. at 1:30 pm, Stephen would have covered the same distance as Tanaya. At 1:30, Tanaya must be 4.5 miles ($= 3 \text{ miles/hr} \times 1.5 \text{ hrs}$) away from the starting point and Stephen would also be 4.5 miles away from the starting point (since he has covered the same distance as Tanaya).

“...and the amount of time it takes Stephen to cover twice the distance that Tanaya has covered?”

I also need to find the time it takes Stephen to cover twice the distance that Tanaya has covered. So, out of his speed of 9 miles/hr, 6 miles/hr will be used to cover twice of what Tanaya will now cover at a speed of 3 miles/hr. The leftover 3 miles/hr ($= 9 \text{ miles/hr} - 6 \text{ miles/hr}$) of his speed will be used to cover an extra 4.5 miles. If you are wondering why he has to cover an extra 4.5 mile, recall that at 1:30 pm, both of them are at a distance of 4.5 miles from the starting point. Stephen should have been at a distance of 9 miles from the starting point if he wants to cover double the distance that Tanaya covers from the starting point.

Hence, now he needs to cover an extra 4.5 miles. At a speed of 3 miles/hr (which we obtained above using $9 \text{ miles/hr} - 6 \text{ miles/hr}$), it will take him 1.5 hrs to cover the extra 4.5 miles ($= 4.5 \text{ miles} / 3 \text{ miles per hour}$). Therefore, at 3:00 pm he would have covered twice the distance that Tanaya would have covered since 12 noon.

The time difference between 1:30 pm and 3:00 pm is 1.5 hrs (90 mins). This is the required time difference.

Answer (C)

How long do you think it takes to think all this?

27.Beyond the rule of thumb

A couple of days back, during a class, we got into a discussion on “If you have ‘n’ variables, you need at least ‘n’ equations to solve for each variable.” It centered on the cases where you cannot apply this rule of thumb. Let me discuss a couple of those cases in detail today:

1. A question such as this:

Question: What is the value of $(x + y + z)$?

(1) $x + 3y - 2z = 4$

(2) $x - y + 4z = 10$

A cursory glance and you might be tempted to mark the answer as (E) and move on. After all, to get the value of $(x + y + z)$, you need the value of x , y and z but you only have two equations so you cannot solve for the three variables so of course the statements are not sufficient, right? Wrong! You don't actually need to solve for x , y and z in this case. If you observe carefully, you will see that when you add the two given equations, you get $2x + 2y + 2z = 14$ giving you $x + y + z = 7$ on simplification. If you had to solve for each of the three variables, then I agree you cannot do it using the two statements. GMAT is replete with such tricks and “exceptional cases.” So use your so called ‘rules’ with care.

2. If you have been preparing for GMAT for a while now, you must have come across a question which is similar to the following:

Question: A boy goes to a supermarket and buys some pencils and erasers. The cost of each pencil is \$0.3 and cost of each eraser is \$0.5. If he bought at least one pencil and at least one eraser and the number of pencils he bought was not four, how many pencils did he buy?

(1) He paid a total of \$4.20

(2) He bought three erasers.

I am sure, by using the first statement, most of you will be able to come up with the following equation: $0.3p + 0.5e = 4.2$ and then, by multiplying both sides by 10, you will get $3p + 5e = 42$

(p – the number of pencils, e – the number of erasers)

What next? This is the equation of a line and has infinite solutions i.e. for every value of p , exists a value of e . For example: When $p = 0$, $e = 42/5$; when $p = 0.1$, $e = 8.34$; when $p = 1$, $e = 39/5$ and so on... Then, should I say that from this statement alone, he can buy the pencils and erasers in an infinite number of ways e.g. 0 pencils and $42/5$ erasers or 0.1 pencils and $41.7/5$ erasers or 1 pencil and $39/5$ erasers etc? Does it mean we need statement 2 as well to get the number of pencils? Actually, no! We don't need the second statement to get our answer. Let's see why.

There are certain constraints to the acceptable solutions. Can I buy $42/5 = 8.4$ erasers? I can buy 8 erasers or 9 erasers but how can I buy 8.4 erasers? So what we are looking for is integral values of p and e . Even though it is not mentioned, our common sense says it has to be so. This is a constraint on possible solutions and will narrow down the acceptable values.

Consider the equation again: $3p + 5e = 42$

One set of integral solutions to this equation is $p = 14$, $e = 0$ (I will discuss how I got this later.) When you put $p = 14$ and $e = 0$ above, you get $3*14 + 5*0 = 42$. Here, $3p = 42$ and $5e = 0$ and they add up to give 42. What if I want to get 42 in another way? I can decrease $3p$ by some amount and will have to increase $5e$ by the same amount to get the same sum of 42 e.g. we can decrease $3p$ by 1 and increase $5e$ by 1 to get $41 + 1 = 42$. So $3p$ was 42, but now we want $3p$ to be 41. What should p be? p should be $41/3$. But this is not an integral value! We are looking for integral solutions only. Then let's try to decrease p instead of $3p$ to ensure that we get integral values of p . If $p = 13$ instead of its previous value of 14, we get $3p = 39$. (We decreased $3p$ by 3.)

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Now, I must increase $5e$ by 3 to get the same sum of 42. $5e$ was 0 and needs to be 3 now. What will e be now? $e = 3/5$. Unfortunately, the problem is still the same. We need integral values of p and e , both. I can increase $5e$ in blocks of 5 only i.e. if $e = 0$, $5e = 0$; $e = 1$, $5e = 5$; $e = 2$, $5e = 10$ etc.

Now the problem is that $3p$ can be decreased only in blocks of 3 and $5e$ can be increased only in blocks of 5. But the decrease in $3p$ has to be offset by the increase in $5e$! Therefore, we should decrease/increase them in blocks of 15 (lowest common multiple of 3 and 5). So when I try to decrease $3p$ by 15, p decreases by 5 (the second term, $5e$, has 5 as the co-efficient) and when I try to increase $5e$ by 15, e increases by 3 (the co-efficient of the first term). The table given below will make it clearer.

<u>$3p$</u>	+	<u>$5e$</u>	=	
($p = 14$) 42		($e = 0$) 0		42
$-5 \downarrow$		$+3 \downarrow$		
($p = 9$) 27		($e = 3$) 15		42
		$\downarrow +15$		
($p = 4$) 12		($e = 6$) 30		42
($p = -1$) -3		($e = 9$) 45		42
($p = 19$) 57		($e = -3$) -15		42
($p = 24$) 72		($e = -6$) -30		42

and so on...

Note that in second, third and fourth rows, we have been decreasing $3p$ and increasing $5e$. We could do the opposite as done in the last two rows of the table above. We could increase p by 5 and decrease e by 3 to get more solutions. Once we have one solution, we can figure out an infinite number of solutions. Then is our answer still infinite with statement (1) alone? Why the heck did we do all this work then? We should have just marked our answer as (C) and moved on.

Actually, there are some other constraints too. Can the number of pencils or erasers be negative? Also, since he buys at least one pencil and at least one eraser, p and e cannot be 0 (so we discard the first solution). Then, a solution is one where values of p and e are positive integers.

Go back to the table. After the third row of solutions, if you keep decreasing $3p$, p will be negative every time. Look at the last row – if you keep decreasing $5e$, e will remain negative. Therefore, there are only two solutions ($p = 9$, $e = 3$) and ($p = 4$, $e = 6$). Since our question stem mentions that p is not equal to 4, we discard the second solution and retain just the first one. This means that statement (1) above is sufficient to get the answer.

Now we come back to 'How do you get the first solution'. Simple – by brute force. Here it is easy since 42 is a multiple of 3. Then we know that p can be 14 to give 42 and $5e$ can be 0.

An equation such as $3p + 5e = 49$ is trickier. So this is how we find a solution:

First, we check for $e = 1$. Reduce 49 by 5 to get 44 and then check – is 44 divisible by 3? – No.

Then check for $e = 2$. Reduce 44 by 5 again to get 39 – is 39 divisible by 3? – Yes!

This means $3p$ can be 39 and $5e$ can be 10 giving us $p = 13$ and $e = 2$. This would be our first solution and would lead us to more, possibly. How many positive integral solutions will this equation have?

Quarter Wit_Quarter Wisdom- Part-1

To sum it up, when we are looking for positive integer solutions, keep in mind that you might have a limited number of values and the constraints given in your question could lead you to a single solution.

Let me leave you here with some other things to ponder upon:

- What if I replace the equation above by $3x + 6y = 40$?
- Should coefficients of x and y be co-prime?
- And, a trickier thing to think about – how many integral solutions would $3x - 5y = 42$ have?

28.The push and the pull around one

Continuing the discussion of some stand alone topics, let's discuss an important number property today. It is not only useful to know for GMAT but, if understood well, will also help you a lot during your MBA, especially if you are keen on subjects in Finance since these subjects use a lot of ratios — e.g., Financial Leverage, P/E etc. You will often come across a situation where you will need to compare ratios. Say, you have a given ratio N/D . Now, a number 'A' is subtracted from both N and D . Is the new ratio $(N - A)/(D - A)$ greater than or less than N/D ? The answer depends on the original value of N/D .

The main concept is as follows:

When we add the same positive integer to the numerator and the denominator of a positive fraction, the fraction increases if it is less than 1 (but remains less than 1) and decreases if it is more than 1 (but remains more than 1). That is, we can say, that the fraction is pulled toward 1 in both the cases.

Let us understand this with the help of some examples:

Example 1: $N/D = 1/2$

If we add 3 to both the numerator and the denominator, N/D changes to $4/5$. Notice that $4/5$ is greater than $1/2$ i.e. it is closer to 1 than $1/2$ but is still less than 1.

Example 2: $N/D = 7/3$

If we add 3 to both the numerator and the denominator, N/D changes to $10/6 = 5/3$. Notice that $5/3$ is less than $7/3$ i.e. it is closer to 1 than $7/3$ but is still greater than 1.

These two examples depict how the ratio is 'pulled' toward 1.

What happens if you subtract the same positive integer (or, in other words, add a negative integer) from both N and D ?

When we subtract the same positive integer from the numerator and the denominator of a positive fraction, the fraction decreases further if it is less than 1 and increases further if it is more than 1. That is, we can say, that the fraction is pushed further away from 1 in both the cases. An assumption here is that the positive number subtracted is less than both the numerator and the denominator.

Example 1: $N/D = 3/5$

If we subtract 1 from both the numerator and the denominator, N/D changes to $2/4 = 1/2$. Notice that $1/2$ is even more less than $3/5$ i.e. it is further away from 1 than $3/5$.

Example 2: $N/D = 7/3$

If we subtract 2 from both the numerator and the denominator, N/D changes to $5/1 = 5$. Notice that 5 is greater than $7/3$ i.e. it is further away from 1 than $7/3$.

These two examples depict how the ratio is 'pushed' further away from 1.

Quarter Wit_Quarter Wisdom- Part-1

Now let's see how this concept helps us in solving GMAT-relevant questions.

Question: Two positive integers that have a ratio of 3:5 are increased in a ratio of 1:1. Which of the following could be the resulting integers?

- (A) 3 and 5
- (B) 5 and 13
- (C) 21 and 30
- (D) 34 and 68
- (E) 75 and 45

When the ratio 3:5 is increased in the ratio 1:1, it means that both the numerator and the denominator are increased by the same number. Since $N/D (=3/5)$ is less than 1, adding the same number to both the numerator and the denominator will take N/D closer to 1 i.e. it will increase N/D but N/D will still remain less than 1. So the answer option should lie between $3/5$ and 1. Only option (C) satisfies this condition and hence answer is (C).

29.The Power of factorials

[Last week](#), we started discussing some number properties. Let's continue that discussion and dive into some more of those. In my opinion, it is the single most important topic on GMAT and one in which the smartest people slip easily. Think of this as a relatively easy way to earn another (or save) 20 or 30 points on your total GMAT score!

Let me show you the concept we will discuss today right away:

QUESTION: If 2^k is a factor of $(10!)$, what is the greatest possible value of k ?

- (A) 5
- (B) 7
- (C) 8
- (D) 10
- (E) 12

SOLUTION:

2^k is a factor of $10!$ We need to find the maximum value of k . This means that we need to find the total number of 2s in $10!$

Let's begin by writing down $10!$ to understand the question: $10! = 1*2*3*4*5*6*7*8*9*10$

Of the numbers above, the following numbers have 2 as a factor: 2, 4 (two of them), 6, 8 (three of them) and 10

Total number of 2s in $10!$ is $1 + 2 + 1 + 3 + 1 = 8$

Easy enough, right? Yes, it is! Problems arise when we are dealing with relatively bigger numbers, say number of 2s in $40!$ or $80!$ etc.

I will now give you a method of solving any such question quickly. Subsequently, I will discuss the logic behind the method.

QUESTION: If 2^k is a factor of $(10!)$, what is the greatest possible value of k ?

METHOD:

Step 1: $10/2 = 5$ (Divide 10 (obtained from $10!$) by 2 (obtained from 2^k))

Step 2: $5/2 = 2$ (Divide 5 (obtained from step 1) by 2 and ignore everything after the decimal)

Step 3: $2/2 = 1$ (Divide 2 (obtained from step 2) by 2)

Now, the quotient obtained in step 3, i.e. 1, is less than the divisor, i.e. 2, hence stop dividing.

Quarter Wit_Quarter Wisdom- Part-1

Step 4: Add all the quotients obtained: $5 + 2 + 1 = 8$

The greatest possible value of k is 8.

LOGIC:

$$10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$$

Each alternate number in the product above will have a 2. Out of 10 numbers, 5 numbers will have a 2. Hence Step 1:
 $10/2 = 5$

These 5 numbers are 2, 4, 6, 8, 10. Each of these numbers give us a 2, therefore we have five 2s as of now.

Now out of these 5 numbers, every alternate number will have another 2 since it will be a multiple of 4. Hence Step 2:
 $5/2 = 2$

These 2 numbers are 4 and 8. Both of these numbers give us another 2, therefore we have two more 2s. Out of these 2 numbers, every alternate number will have yet another 2 because it will be a multiple of 8. Hence Step 3: $2/2 = 1$

This single number is 8. It gives us one more 2.

Now, all 2s are accounted for. Just add them $5 + 2 + 1 = 8$ (Hence Step 4)

These are the number of 2s in $10!$

Similarly, you can find maximum power of any prime number in any factorial.

Let's quickly run through a few questions to grasp the concept properly.

Question: If 3^k is a factor of $(122!)$, what is the greatest possible value of k ?

Solution:

Step 1: $122/3 = 40$

Step 2: $40/3 = 13$

Step 3: $13/3 = 4$

Step 4: $4/3 = 1$

Step 5: $40 + 13 + 4 + 1 = 58$

Greatest possible value of k is 58.

Question: If 4^k is a factor of $(122!)$, what is the greatest possible value of k ?

Solution:

4 is not a prime number but to get the number of 4s in $122!$, we can find the number of 2s and half it. (Mind you, it is not the same as finding the number of 4s. Think 'Why?')

Step 1: $122/2 = 61$

Step 2: $61/2 = 30$

Step 3: $30/2 = 15$

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Step 4: $15/2 = 7$

Step 5: $7/2 = 3$

Step 6: $3/2 = 1$

Step 7: $61 + 30 + 15 + 7 + 3 + 1 = 117$

Total number of 2s is 117 so total number of 4s is 58. Greatest possible value of k is 58.

Question: If 6^k is a factor of $(40!)$, what is the greatest possible value of k?

Solution:

6 is not a prime number but we make a 6 by multiplying 2 with 3. To get the number of 6s in $40!$, we just need to find the number of 3s because the number of 3s will be fewer than the number of 2s. If you are a little confused, don't worry. Look at the solution given below.

Let's find the number of 2s in $40!$

Step 1: $40/2 = 20$

Step 2: $20/2 = 10$

Step 3: $10/2 = 5$

Step 4: $5/2 = 2$

Step 5: $2/2 = 1$

Step 6: $20 + 10 + 5 + 2 + 1 = 38$

Total number of 2s is 38.

Let's find the number of 3s now.

Step 1: $40/3 = 13$

Step 2: $13/3 = 4$

Step 3: $4/3 = 1$

Step 4: $13 + 4 + 1 = 18$

Total number of 3s is 18.

To make a 6, you need one 2 and one 3. In $40!$, you have 38 2s but only 18 3s. So you can make only 18 6s. Therefore, maximum value of k is 18. It is obvious that the total number of a higher prime will be less than the total number of a smaller prime. Hence, you don't even need to find the number of 2s here.

Usually, the greatest prime number will be the limiting condition, but not always. Think what happens when we need to find the greatest power of 12 in $40!$

30.Exponents- The Bane of your preparation

No matter what score you are aiming for, knowing how to deal with exponents is crucial. Generally, questions based solely on exponents lie in the range of 600-650 and are one of the favorite topics of GMAT (but that's just my opinion). It is also a topic which is very easy once you get the 'hang of it' but generates instant dislike until you don't. I am sure

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many of you hate the sordid looking expressions/equations such as

$$(2^a \times 4 \times 3^{a-4} \times 3^b) / (3^4 \times 2^2) = 8^{a-4} \times 729$$

But of course, I don't blame you. Sadly, exponents and roots are basics that we should be experts in – whether we are working on Number Systems, Algebra or even Geometry! So let's roll!

Today I will just discuss the basic rules for people who are averse to exponents (I will try and explain these rules in detail to help them get comfortable). For roots and tricky questions, watch out for the next posts.

Some Basic Mathematics:

$3^4 = 3 \times 3 \times 3 \times 3$ (When a number is multiplied by itself, you raise it to a power)

$3 \times 4 = 3 + 3 + 3 + 3$ (When a number is added to itself, you multiply. In essence, multiplication is just addition. But that's a topic for another day.)

In the first case above, the '3' is called the base and the '4' is called the exponent.

It follows then that 3^2 would be just 3×3 . So now, if we multiply 3^4 by 3^2 , what do we get? $(3 \times 3 \times 3 \times 3) \times (3 \times 3) = 3^6$. The indices just got added!

Rule 1: $a^m \times a^n = a^{(m+n)}$

When you multiply two numbers that have the same base, their exponents get added.

Example: $4^5 \times 4^3 = ?$

$$4^5 \times 4^3 = 4^{(5+3)} = 4^8$$

Now tell me, what happens when we divide 3^4 by 3^2 ?

$$3^4 / 3^2 = (3 \times 3 \times 3 \times 3) / (3 \times 3) = 3 \times 3$$

But we know that $3 \times 3 = 3^2$. So essentially, $3^4 / 3^2 = 3^{(4-2)} = 3^2$

Rule 2: $a^m / a^n = a^{(m-n)}$

When you divide two numbers that have the same base, the exponent of the divisor gets subtracted from the exponent of the dividend.

Example: $4^5 / 4^3 = ?$

$$4^5 / 4^3 = 4^{(5-3)} = 4^2$$

Let's look at another important case.

$$4^4 = (2^2)^4 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (2 \times 2) = 2^8$$

The exponents just got multiplied!

Rule 3: $(a^m)^n = a^{mn}$

Example: $4^5 \times 2^8 = ?$

$$4^5 \times 2^8 = (2^2)^5 \times 2^8 = 2^{10} \times 2^8 = 2^{(10+8)} = 2^{18}$$

Rule 4: For any number a, $a^0 = 1$

Example: $3^0 = 1$

Now that we have discussed Multiplication and Division, we need to think about Addition and Subtraction.

Quarter Wit_Quarter Wisdom- Part-1

What happens when we add 3^5 to 3^3 ? What is $3^5 + 3^3$? Can I still add the exponents? Think. $(3 \times 3 \times 3 \times 3 \times 3) + (3 \times 3 \times 3) = 243 + 27 = 270$. We cannot play with the exponents when dealing with Addition and Subtraction. But we can take a common factor. Let me show you:

Example: $3^5 + 3^3 = ?$

$$3^5 + 3^3 = 3^3 \times (3^2 + 1) = 27 \times 10 = 270$$

Here, we have taken three 3s out from both the numbers and added the rest. It saves us time. We don't need to calculate $3 \times 3 \times 3 \times 3 \times 3$. Similarly, we can handle subtraction too.

Let's take a quick question now.

Question:

Given $(2^n \times 4 \times 2^4)/8 = 1$, what is the value of n?

Solution:

We know that 4 is 2^2 and 8 is 2^3 .

$$\text{We get: } (2^n \times 2^2 \times 2^4)/2^3 = 1$$

Here, all the terms have the same base i.e. 2 and the terms are multiplied/divided. Therefore, the exponents can be added/subtracted.

$$2^{(n+2+4)} / 2^3 = 1$$

$$2^{(n+2+4-3)} = 2^{(n+3)} = 1$$

$$2^{(n+3)} = 2^0$$

Wait a minute! From where did we get 2^0 ? Since we know that $2^0 = 1$, if we have 1, we can write it as 2^0 . A term with base 2 will be equal to 1 only if the exponent is 0.

Now, since the bases on both sides of the equation are same, the exponents should also be the same.

$$n + 3 = 0$$

$$n = -3$$

Note: We used the long method to solve this question since we wanted to discuss the application of various rules. You can use faster approaches once you are comfortable with these basic rules.

More on negative exponents in the next post.

31.Exponents- The bane of your Preparation-II

Let me recap the rules we discussed in the last post:

$$\text{Rule 1: } a^m \times a^n = a^{(m+n)}$$

$$\text{Rule 2: } a^m / a^n = a^{(m-n)}$$

$$\text{Rule 3: } (a^m)^n = a^{mn}$$

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Rule 4: For any number a , $a^0 = 1$

In Rules 1 and 2 above, the bases were the same. What happens if the exponents are the same but the bases are different?

$$2^4 \times 3^4 = (2 \times 2 \times 2 \times 2) \times (3 \times 3 \times 3 \times 3) = (2 \times 3) \times (2 \times 3) \times (2 \times 3) \times (2 \times 3) = 6^4$$

The bases got multiplied!

Rule 5: $a^m \times b^m = (a \times b)^m$

which also implies that $(a \times b)^m = a^m \times b^m$

Example:

$$6^5 = 2^5 \times 3^5$$

The treatment of division is very similar.

$$6^4 / 2^4 = (6 \times 6 \times 6 \times 6) / (2 \times 2 \times 2 \times 2) = 3 \times 3 \times 3 \times 3 = 3^4$$

Essentially, the bases get divided and the exponent remains the same.

Rule 6: $a^m / b^m = (a / b)^m$

which also implies that $(a / b)^m = a^m / b^m$

Example:

$$(3/2)^5 = 3^5 / 2^5$$

Let's take a quick look at negative exponents now.

What is 3^{-4} ? It is extremely easy to handle negative exponents. Just flip the number and the exponent becomes positive. So 3^{-4} is $1/3^4$. This also implies the following:

- $1/3^{-4} = 3^4$
- $3^4 = 1/3^{-4}$
- $1/3^4 = 3^{-4}$

When you want to change the sign of the exponent, flip the number. When you want to flip the number, change the sign of the exponent!

Example: What is the value of $3^2 / 3^{-3}$

This can be solved in two ways:

1. Flip the number to get rid of the negative exponent.

Quarter Wit_Quarter Wisdom- Part-1

$$3^2 / 3^{-3} = 3^2 * 3^3$$

The base is the same and the two are multiplied so we add the exponents.

$$3^2 * 3^3 = 3^{(2+3)} = 3^5$$

OR

2. The bases are the same and the terms are divided so subtract the exponent of the divisor from the exponent of the dividend.

$$3^2 / 3^{-3} = 3^{(2 - (-3))} = 3^5$$

Now that we have covered all the major rules of exponents, let's work on the question we saw in the previous post.

Question: If $(2^a \times 4 \times 3^{-4} \times 3^b) / (3^4 \times 2^2) = 8^{-4} \times 729$, what is the value of a and b?

- (A) -10 and -14
- (B) 10 and 12
- (C) -10 and 12
- (D) -12 and 14
- (E) 12 and -14

Solution: First, we need to bring everything to prime number form.

$$(2^a \times 4 \times 3^{-4} \times 3^b) / (3^4 \times 2^2) = 8^{-4} \times 729$$

$$(2^a \times 2^2 \times 3^{-4} \times 3^b) / (3^4 \times 2^2) = (2^3)^{-4} \times 3^6$$

If you do not remember that 729 is the sixth power of 3, you should know that it will be some power of 3 because the left hand side of the equation has only two prime numbers – 2 and 3. So the right hand side of the equation cannot have any prime other than 2 and 3 (if there is some other prime number, its exponent will be 0 to make the term 1). Since 729 is certainly not a power of 2 (since it is odd), it must be some power of 3. We just need to multiply 3 with itself a few times to figure out the power.

Let's work on the left hand side of the equation first. Get rid of negative exponent.

$$(2^a \times 2^2 \times 3^b) / (3^4 \times 3^4 \times 2^2)$$

Some terms have same bases and are multiplied. Add their exponents.

$$(2^{(a+2)} \times 3^b) / (3^8 \times 2^2)$$

Some terms have same bases and are divided. Subtract their exponents.

Quarter Wit_Quarter Wisdom- Part-1

$$2^a \times 3^{(b-8)}$$

Let's equate this to the right hand side now.

$$2^a \times 3^{(b-8)} = (2^3)^4 \times 3^6$$

Using Rule 3 on the right hand side,

$$2^a \times 3^{(b-8)} = 2^{(-12)} \times 3^6$$

The exponent of 2 on left hand side is 'a' and on right hand side is -12. Therefore, $a = -12$.

The exponent of 3 on left hand side is $(b-8)$ and on right hand side is 6. Therefore, $b-8 = 6$ or $b = 14$

Answer (D)

Note: We used the long method to solve this question since we wanted to discuss the application of various rules. You can use faster approaches once you are comfortable with these basic rules.

Let me leave you with a couple of questions. I will discuss these in the next post.

Question 1: Given $(1/4)^{18} \times (1/5)^n = 1/(2 \times 10^{35})$, find the value of n.

Question 2: Is $5^m < 1000$?

Statement 1: $5^{(m+1)} > 3000$

Statement 2: $5^{(m-1)} = 5^m - 500$

32.The theory of exponents applied

As promised, in this post, I will discuss the questions I gave you in [the last post](#). Let's apply the rules of exponents that we have learned. I will recap all the rules first and then we will proceed to the questions.

$$\text{Rule 1: } a^m \times a^n = a^{(m+n)}$$

$$\text{Rule 2: } a^m / a^n = a^{(m-n)}$$

$$\text{Rule 3: } (a^m)^n = a^{mn}$$

$$\text{Rule 4: For any number } a, a^0 = 1$$

$$\text{Rule 5: } a^m \times b^m = (a \times b)^m$$

which also implies that $(a \times b)^m = a^m \times b^m$

$$\text{Rule 6: } a^m / b^m = (a / b)^m$$

Quarter Wit_Quarter Wisdom- Part-1

which also implies that $(a/b)^m = a^m/b^m$

Question 1: Given $(1/4)^{18} \times (1/5)^n = 1/(2 \times 10^{35})$, find the value of n.

In this question, we have 4s and 5s on the left hand side and 2s and 10s on the right hand side. How will we equate the exponents if the bases are different? We cannot equate the exponents in that case. So what we need to do here is make bases same. Let's bring down all bases to prime number form.

$$(1/4)^{18} \times (1/5)^n = 1/(2 \times 10^{35})$$

$$(1/2^2)^{18} \times (1/5)^n = 1/(2 \times (2 \times 5)^{35})$$

(I suggest you to write down these steps in your notebook. Since I cannot format the numbers properly in the editor, the above looks confusing even though it is very straight forward.)

Using Rule 6: $(a/b)^m = a^m/b^m$ on left hand side, we get

$$1^{18}/(2^2)^{18} \times 1^n/5^n = 1/(2 \times (2 \times 5)^{35})$$

1 to any power is 1. Next we use Rule 5: $(a \times b)^m = a^m \times b^m$ on right hand side to get

$$1/(2^2)^{18} \times 1/5^n = 1/(2 \times 2^{35} \times 5^{35})$$

When we write 2, it implies that the power here is 1 i.e. $2 = 2^1$. We substitute this on right hand side and use Rule 3: $(a^m)^n = a^{mn}$ on left hand side to get

$$1/2^{36} \times 1/5^n = 1/(2^1 \times 2^{35} \times 5^{35})$$

Now we use Rule 1: $a^m \times a^n = a^{(m+n)}$ on right hand side

$$1/2^{36} \times 1/5^n = 1/(2^{36} \times 5^{35})$$

Notice that the power of 2 is the same on left and right hand side (as expected). The power of 5 on the left hand side is n and on the right hand side it is 35. For the equation to hold, n must be 35.

(Or you could have noticed right in the beginning that the only 5 on the left hand side is the one which is raised to the power of n and on the right hand side, you will get 5^{35} since you will get 5 only from 10^{35} .)

Question 2: Is $5^m < 1000$?

Statement 1: $5^{(m+1)} > 3000$

Statement 2: $5^{(m-1)} = 5^m - 500$

Solution:

First note that $5^4 = 625$ and $5^5 = 3125$ (even if you do not know this, it is fine. You don't need to calculate. Just observe that $5^5 = 625 \times 5$ will be greater than 3000 since $600 \times 5 = 3000$)

Statement 1: $5^{(m+1)} > 3000$

This means $m+1$ is greater than 4 which implies that m is greater than 3. It doesn't mean that $m+1$ is at least 5 because the question doesn't say that m has to be an integer. $m+1$ could be 4.999 making $m = 3.999$. Since m can take values less than 4 and more than 4, 5^m could be less than 1000 and more than 1000. Hence this statement is not sufficient to say whether 5^m is less than 1000.

Statement 2: $5^{(m-1)} = 5^m - 500$

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We re-arrange the given equation to get: $500 = 5^m - 5^{(m-1)}$. Notice that on the right hand side, the terms are subtracted. So you cannot do anything with the exponents except take something common. What can you take common from 5^m and $5^{(m-1)}$? Obviously, $5^{(m-1)}$. This is not intuitive to many people. Let me show you an example first.

Say, the given equation is $4 \times 5^a = 5^7 - 5^6$

What can you take common from the right hand side? I think most of you will agree it is 5^6 i.e. the term with the smaller exponent. The right hand side will become $5^6 (5 - 1) = 4 \times 5^6$ so that we will get $a = 6$. Similarly, when you have the equation $500 = 5^m - 5^{(m-1)}$, what can you take common from the right hand side? You can take $5^{(m-1)}$ i.e. the term with the smaller exponent. Also, $500 = 125 \times 4 = 5^3 \times 4$. So the equation becomes:

$$5^3 \times 4 = 5^{(m-1)} \times (5 - 1)$$

$$5^3 = 5^{(m-1)} \text{ (Cancel 4 from both sides)}$$

Therefore, $m - 1 = 3$ giving us $m = 4$

Hence, $5^m = 625$ which is less than 1000. This statement is sufficient to tell us that 5^m is less than 1000.

Answer (B).

These are some applications of Exponents that you need to understand well in order to comfortably work with exponents on the GMAT. There are many more interesting concepts and questions related to exponents so start practicing!

33.Are there calculation intensive questions in GMAT

thought I will take up "Roots" today but the rules of exponents and roots can get pretty monotonous. So let's take a break today and do something more interesting. I keep telling my students that GMAT questions do not involve long calculations. If you find yourself dividing a three digit number with a two digit number, it means you have missed the point of the question. Today, I will depict what I mean with the help of a couple of examples. I start with one my favorite questions from the Veritas book.

Question 1: A certain portfolio consisted of 5 stocks, priced at \$20, \$35, \$40, \$45 and \$70, respectively. On a given day, the price of one stock increased by 15%, while the price of another decreased by 35% and the prices of the remaining three remained constant. If the average price of a stock in the portfolio rose by approximately 2%, which of the following could be the prices of the shares that remained constant?

- (A) 20, 35, 70
- (B) 20, 45, 70
- (C) 20, 35, 40
- (D) 35, 40, 70
- (E) 35, 40, 45

What we notice here is that one stock price increases by 15% and another decreases by 35% but still, there is an overall INCREASE of 2%. Hence, the amount of increase should be greater than the amount of decrease. Say, the stock whose price increases by 15% is A and the stock whose price decreases by 35% is B. We can infer that $15\% \text{ of } A > 35\% \text{ of } B$

Now an interesting point is that 15% of A will be equal to 30% of B if A is twice of B. But 15% of A is greater than 30% of B so A must be greater than twice of B. In fact 15% of A is greater than 35% of B so A must be substantially greater than twice of B. Can we infer something about the value of B from this? Can we say that A, B is a pair of numbers such that A is more than twice of B? Out of 20, 35, 40, 45 and 70, what can be the value of B? B has to be 20 because we have values more than twice of 20 (which are 45 and 70). But we don't have any values which are more than twice of 35, 40, 45 and 70.

Quarter Wit_Quarter Wisdom- Part-1

If B is 20, A must be either 45 or 70 (since A must be more than twice of B). I would bet on 70 since A has to be substantially greater than twice of 20. But say, there is this little voice inside you that is constantly telling you to be extra careful before marking the answer. After all, there is no partial credit. Only the final answer is what matters and if you slip up in the last step, it's as bad (actually worse since you used up the time allotted to this question) as being totally bowled over by the question (in such a case, you should move on within 30 secs and save the time for another, more do-able problem).

Let's do some quick calculations to confirm that the correct answer is indeed 70 and not 45:

10% of 20 is 2.

30% of 20 is 6.

5% of 20 is 1.

Therefore, 35% of 20 is $6+1 = 7$.

15% of A has to be greater than 7.

70 satisfies this condition – 10% of 70 is 7 so 15% of 70 must be greater than 7.

45 does not satisfy this condition – 10% of 45 is 4.5 and 5% of 45 is 2.25, therefore, 15% of 45 is 6.75 which is less than 7

We see that the stock prices that did not change are 35, 40 and 45. Hence the answer is (E).

The question might have seemed intimidating at first but wouldn't you say that it looks much more reasonable now? Let's try another question (from some outside source) which looks calculation intensive, but in fact, involves no calculations at all.

Question 2: Which of the following, when squared, will yield a value greater than $\frac{3}{4}$

- (A) $\frac{2}{7}$
- (B) $(.75)^2$
- (C) $\frac{2}{3}$
- (D) $\frac{6}{7}$
- (E) $\frac{7}{8}$

You might think that you need to square each option and divide to find out the answer but don't get alarmed just yet. Let's give 'logic' a try first. We know that when a positive number less than 1 is squared, it becomes even smaller.

For example:

$(\frac{1}{2})^2 = \frac{1}{4}$. $\frac{1}{4}$ is smaller than $\frac{1}{2}$

$(\frac{2}{3})^2 = \frac{4}{9}$. $\frac{4}{9}$ is smaller than $\frac{2}{3}$. If you are wondering how to figure this out, $\frac{4}{9}$ is less than $\frac{1}{2}$ and $\frac{2}{3}$ is greater than $\frac{1}{2}$. Or multiply and divide $\frac{2}{3}$ by 3 to get $\frac{6}{9}$. $\frac{4}{9}$ is less than $\frac{6}{9}$

All the given options are less than 1. When you square them, they will become even smaller.

So the answer must be greater than $\frac{3}{4}$ to begin with and must be much greater than $\frac{3}{4}$ so that even after squaring, it remains greater than $\frac{3}{4}$. Let's consider the individual options.

Option (A) $\frac{2}{7} < \frac{3}{4}$ – Ignore

Option (B) $(.75)^2 < .75$ (which is $\frac{3}{4}$) – Ignore

Option (C) $\frac{2}{3} < \frac{3}{4}$ – Ignore

Quarter Wit_Quarter Wisdom- Part-1

6/7 and 7/8 are both greater than 3/4. If 6/7 were the answer and the square of 6/7 were greater than 3/4, since 7/8 is even greater than 6/7, its square would be greater than 3/4 too. But we cannot have multiple answers. Hence, only the square of 7/8 must be greater than 3/4. Answer must be 7/8.

Or you can just consider that each option is less than 1. We need the option which is greater than 3/4 when squared. So the answer would be the option which is the largest since only one correct answer is there and one correct answer is definitely there!

I hope these examples showed you how you can deal with most questions without doing any long calculations. More on this, later...

You are welcome to put up calculation-intensive questions in the comments and we at Veritas will try and provide 'logical' solutions. We don't promise that there are calculation-cutting tricks for all questions (especially the harder Permutation Combination questions) since sometimes, some providers give long calculations only to make the questions harder – they totally miss the point – but in most cases, we would be able to do justice.

34.Tackling the Roots

First of all, understand that roots are a subset of exponents i.e. any number with any root can be converted to an exponent. Every rule of exponent that we have learnt till now will then be applicable.

$$\text{Rule 1: } \sqrt[n]{a} = a^{\frac{1}{n}}$$

$$\text{Example 1: } \sqrt[3]{8} = 8^{\frac{1}{3}}$$

Let's see this rule in action.

$$\text{Question 1: What is } \sqrt[3]{9} \times \sqrt[3]{3} ?$$

Solution:

$$\sqrt[3]{9} = 9^{\frac{1}{3}} = (3^2)^{\frac{1}{3}} = 3^{\frac{2}{3}} \text{ (Using Exponents Rules we learned in the previous posts)}$$

$$\sqrt[3]{3} = 3^{\frac{1}{3}}$$

$$\sqrt[3]{9} \times \sqrt[3]{3} = 3^{\frac{2}{3}} \times 3^{\frac{1}{3}} = 3^{\frac{2}{3} + \frac{1}{3}} = 3^1 = 3$$

The rules for exponents are the same whether we have an integer as the exponent or a fraction. Therefore, if you think that roots are a nightmare but exponents are fine, just convert the roots to exponents form and the question becomes utterly do-able. But, when you take that route, it takes far too much extra time. So, it makes sense to review the rules of roots (which are very similar to the rules of exponents)

$$\text{Question 2: What is } \sqrt{4} ?$$

We know this is 2.

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How?

$$\sqrt{4} = 4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} = 2$$

Or we can directly say that there are two 2s under the square root sign. When we remove the square root sign, we are left with a single 2.

$$\sqrt{4} = \sqrt{2 \times 2} = 2$$

Question 3: On the same lines, what is $\sqrt[3]{27}$?

$$\sqrt[3]{27} = \sqrt[3]{3 \times 3 \times 3} = 3$$

Since we need the third root here, the three 3s give us a single 3.

Question 4: What is $\sqrt[4]{64}$?

$$\sqrt[4]{64} = \sqrt[4]{2 \times 2 \times 2 \times 2 \times 2 \times 2} = 2\sqrt[4]{4}$$

Here, we need the fourth root of 64. 64 consists of six 2s. Since this is the fourth root, out of four 2s, we take one 2 out. The rest of the two 2s remain put.

Question 5: What is $\sqrt[3]{432}$?

$$\sqrt[3]{432} = \sqrt[3]{2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3} = 2 \times 3 \times \sqrt[3]{2} = 6\sqrt[3]{2}$$

432 has four 2s and three 3s. Since we need the third root, three 2s are used to get one 2 out, three 3s are used to get one 3 out and a 2 is leftover under the third root.

I hope this method is clear. Let's go on now.

$$\text{Rule 2: } \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{a \times b}$$

$$\text{Example 1: } \sqrt{2} \times \sqrt{3} = \sqrt{6}$$

$$\text{Example 2: } \sqrt[3]{7} \times \sqrt[3]{49} = \sqrt[3]{7 \times 7 \times 7} = 7$$

$$\text{Rule 3: } \sqrt[n]{a} / \sqrt[n]{b} = \sqrt[n]{a/b}$$

$$\text{Example 1: } \sqrt{8} / \sqrt{2} = \sqrt{4} = 2$$

$$\text{Example 2: } \sqrt[3]{49} / \sqrt[3]{7} = \sqrt[3]{7}$$

$$\text{Rule 4: } \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Quarter Wit_Quarter Wisdom- Part-1

Example 1: $\sqrt[4]{\sqrt{256}} = \sqrt[8]{256} = \sqrt[8]{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} = 2$

Let's look at a complicated question now.

Question 6: Given that $\frac{\sqrt[8]{512 \times (\sqrt{343})^3}}{\sqrt{4} \times \sqrt[3]{\sqrt[2]{128}}} = 2^a \times 7^b$, what is the value of a and b?

(A) $a = 1, b = 1/3$

(B) $a = 12/17, b = 2/3$

(C) $a = -10/21, b = 11/2$

(D) $a = 11/24, b = 4/9$

(E) $a = -25/24, b = 9/2$

Let's focus on the left hand side of the equation first.

$$\frac{\sqrt[8]{512 \times (\sqrt{343})^3}}{\sqrt{4} \times \sqrt[3]{\sqrt[2]{128}}}$$

Using Rule 4: $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$, we get

$$\frac{\sqrt[8]{512 \times (\sqrt{343})^3}}{\sqrt{4} \times \sqrt[3]{\sqrt[2]{128}}} = \frac{\sqrt[8]{512 \times (\sqrt{343})^3}}{\sqrt{4} \times \sqrt[6]{128}}$$

We have some big numbers here. Let's bring them down to prime factors.

$$512 = 2^9$$

$$343 = 7^3$$

$$128 = 2^7$$

$$\frac{\sqrt[8]{512 \times (\sqrt{343})^3}}{\sqrt{4} \times \sqrt[3]{\sqrt[2]{128}}} = \frac{\sqrt[8]{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times (\sqrt{7 \times 7 \times 7})^3}}{\sqrt{4} \times \sqrt[6]{2 \times 2 \times 2 \times 2 \times 2 \times 2}}$$

$$\frac{\sqrt[8]{512 \times (\sqrt{343})^3}}{\sqrt{4} \times \sqrt[3]{\sqrt[2]{128}}} = \frac{2 \times \sqrt[8]{2} \times (7\sqrt{7})^3}{2 \times 2 \times \sqrt[6]{2}}$$

The 2 in the numerator gets canceled with the 2 in the denominator.

$$\frac{\sqrt[8]{512 \times (\sqrt{343})^3}}{\sqrt{4} \times \sqrt[3]{\sqrt[2]{128}}} = \frac{\sqrt[8]{2} \times (7\sqrt{7})^3}{2 \times \sqrt[6]{2}}$$

Quarter Wit_Quarter Wisdom- Part-1

Using Rule 1, let's convert the rest of the roots to exponents.

$$\frac{\sqrt[8]{512} \times (\sqrt{343})^3}{\sqrt{4} \times \sqrt[3]{\sqrt[2]{128}}} = \frac{2^{\frac{1}{8}} \times 7^{\frac{3}{2}} \times 7^{\frac{3}{2}}}{2^{\frac{1}{2}} \times 2^{\frac{1}{6}}}$$

Now, using the rules of exponents, we further simplify this to get

$$\frac{\sqrt[8]{512} \times (\sqrt{343})^3}{\sqrt{4} \times \sqrt[3]{\sqrt[2]{128}}} = \frac{2^{\frac{1}{8}} \times 7^{\frac{3}{2} + \frac{3}{2}}}{2^{\frac{1}{2} + \frac{1}{6}}} = 2^{\frac{1}{8} - \frac{7}{6}} \times 7^{\frac{3}{2} + \frac{3}{2}}$$

$$\frac{\sqrt[8]{512} \times (\sqrt{343})^3}{\sqrt{4} \times \sqrt[3]{\sqrt[2]{128}}} = 2^{-\frac{25}{24}} \times 7^{\frac{9}{2}}$$

Therefore, a = -25/24 and b = 9/2

Answer (E)

Even though it looks really complicated, I would suggest you to go step by step. Just use the rules we have learned and you will get to the answer.

35.Comparing Roots & Exponents

I guess the posts on Exponents and Roots have been lifeless for the most part. If I thought they were a drag to write (I hate rules!) , I am sure you thought they were a torture to read. Nevertheless, if you find yourself wondering what to do when you see $(3^2 \times 9^4)$, you need to go through them (can't live without the rules either!) The good news is that now that we are done with the basics, we can go on to the more "fun" concepts. And the best way to learn fun concepts is through fun questions. So let's take a couple of problems (found them on a [GMAT forum](#)) dealing with comparisons of exponents/roots.

Question 1: Which of the following represents the greatest value?

- A) $2^{2/3} + 3^{3/4} + 4^{4/5} + 5^{5/6}$
- B) $2/3 + 3/4 + 4/5 + 5/6$
- C) $2^{2/3^2} + 3^{2/4^2} + 4^{2/5^2} + 5^{2/6^2}$
- D) $1 - 1/3 + 4/5 - 3/4$
- E) $1 - 3/4 + 4/5 + 1/3$

Solution:

My first thought is to eliminate some options. I see options D and E have negative terms. A quick look tells me that they are definitely smaller than the other options.

Option D: $1 - 1/3 + 4/5 - 3/4$

$(1 - 1/3)$ is $2/3$ and $(4/5 - 3/4)$ is something very small too. The sum of these two terms is definitely much less than the sum obtained in option B where each term is a little less than 1. So option D cannot be the answer.

Option E: $1 - 3/4 + 4/5 + 1/3$

Quarter Wit_Quarter Wisdom- Part-1

$(1 - 3/4)$ is $1/4$

$1/4 + 4/5 + 1/3$ is definitely less than the sum obtained in option B. So option E cannot be the answer.

Option C: Each term of option B is squared in this option. Each term of option B is less than 1 so when you square it, it becomes even smaller (concept was discussed in [this post](#)). Hence every term of option C is smaller than every corresponding term of option B. Therefore, the sum obtained in option C will be less than the sum obtained in option B.

Now we are left with two options, A and B. Let's compare them. You know that we can easily compare fractions that have the same numerator or denominator. For example, out of $2/7$ and $4/7$, we know that $4/7$ is greater because it has the greater numerator. Out of $4/9$ and $4/5$, we know that $4/5$ is greater because it has the smaller denominator. How do you compare when both numerator and denominator are different? Simple – You need to make either their denominator or numerator equal.

Say I want to compare $2/3$ with $2/3$.

I just multiply and divide $2/3$ by 3 to get $6/3$.

Since 6 , the numerator of $6/3$, is greater than 2 , the numerator of $2/3$, we get that $2/3$ is greater than $2/3$.

Similarly, all terms of option A will be greater than all corresponding terms of option B. Therefore, the sum obtained in option A will be more than the sum obtained in option B.

Answer (A).

Note: You can say intuitively that $2/3$ is greater than $2/3$ because 2 is not much smaller than 2 and 3 is not much smaller than 3 but the difference between 2 and 3 is much smaller than the difference between 2 and 3 . But you need to know your numbers very well to make such intuitive decisions correctly. If you have any doubts, just follow the approach of comparing by making the numerator/denominator equal.

Let's look at another example now. This one asks us to compare roots.

Question 2: Which of the following quantities is the largest?

(A) square root (2)

(B) cube root (3)

(C) fourth root (4)

(D) fifth root (5)

(E) sixth root (6)

Solution:

First, let's convert the roots to exponents.

We have to find the greatest out of: $2^{(1/2)}$, $3^{(1/3)}$, $4^{(1/4)}$, $5^{(1/5)}$ and $6^{(1/6)}$

Since fractional powers are a pain, let us multiply all the powers by 60 (the LCM of 2, 3, 4, 5 and 6) to make them manageable. Note here that even though we have changed the numbers, we can still compare them. It is like saying that we have two numbers a and b , both positive and greater than 1. If we find out which of a^{60} and b^{60} is greater, we can find whether a is greater than b or not. The logic is that if $a > b$ (given a and b are greater than 1), then any positive integral power of a will be greater than the same power of b i.e. $a^{20} > b^{20}$, $a^{27} > b^{27}$, $a^{60} > b^{60}$ etc. We are using the same logic here.

The numbers now become: 2^{30} , 3^{20} , 4^{15} , 5^{12} and 6^{10} .

Let's compare these. First thing we notice is that $4^{15} = (2^2)^{15} = 2^{30}$. So option A and C have the same value. Since there is only one answer, we can be certain that it is not out of A and C. There must be another value greater than 2^{30} .

We are left with: 3^{20} , 5^{12} and 6^{10} .

Bases and powers, both are different in these three options. To compare, we need to make one of them the same. We see that $3^{20} = 9^{10}$. We can compare this with 6^{10} . We see that 9^{10} (i.e. 3^{20}) is greater.

Now we just need to compare 3^{20} with 5^{12} .

$$3^{20} = (3^5)^4 = 243^4$$

$$5^{12} = 125^4$$

Quarter Wit_Quarter Wisdom- Part-1

Out of these two, 3^{20} is greater i.e. $3^{(1/3)}$ is the greatest of all the five options.

Answer (B)

A little bit of manipulation in both the questions led us quickly to the answers. Hope you enjoyed working on these problems.

Keep practicing!

36.Analyzing 700+ GMAT Problems

This week, let's look at some more properties of exponents and roots. Using a high level data sufficiency question, we will see how a number x is related to $\sqrt{x}$ and to x^3 .

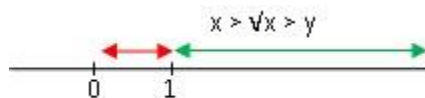
Question: Is $x > y$?

Statement 1: $\sqrt{x} > y$

Statement 2: $x^3 > y$

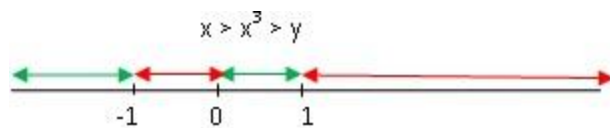
It is one of those gorgeous questions that seem very simple at first but surprise you later.

The question asks us whether x is greater than y , but the statements tell us the relation between $\sqrt{x}$, x^3 and y . If we know that $\sqrt{x}$ is greater than y , when can we say that x is certainly greater than y too? If we know that x is greater than $\sqrt{x}$, then we can say for sure that x is greater than y too. Is x always greater than $\sqrt{x}$? No. Look at the diagram given below.



$\sqrt{x}$ is not defined for negative values of x so let's ignore the section to the left of 0. When the value of x lies between 0 and 1, x is less than $\sqrt{x}$ (for example: when $\sqrt{x} = 1/2$, $x = 1/4$). When the value of x is greater than 1, x is greater than $\sqrt{x}$ (for example: when $\sqrt{x} = 2$, $x = 4$).

Similarly, let's look at the relation between x^3 and x . If we know that x^3 is greater than y , when can we say that x is certainly greater than y too? If we know that x is greater than x^3 , then we can say for sure that x is greater than y too. Is x always greater than x^3 ? No. Look at the graph below.



When the value of x lies between -1 and 0 or in the region greater than 1, x is less than x^3 (for example: when $x = 2$, $x^3 = 8$). When the value of x lies in the region less than -1 or between 0 and 1, x is greater than x^3 (for example: when $x = 1/2$, $x^3 = 1/8$).

Let's look at the statements now

Statement 1: $\sqrt{x} > y$

Since $\sqrt{x}$ is not defined for negative x , we get that $x \geq 0$. As we saw in the first graph above, for some values, x is greater than $\sqrt{x}$, for others, x is less than $\sqrt{x}$. When x is less than $\sqrt{x}$, x may not be greater than y . So this statement alone is not sufficient.

Statement 2: $x^3 > y$

Quarter Wit_Quarter Wisdom- Part-1

As we saw in the second graph above, for some values, x is greater than x^3 , for others, x is less than x^3 . When x is less than x^3 , x may not be greater than y . So this statement alone is not sufficient.

Using both the statements together, we know that $x \geq 0$.

When x lies between 0 and 1, we know that $x \geq x^3$. Since statement (2) says that $x^3 > y$, we can say that $x > y$.

When x is greater than 1, we know that $x > x^3$. Since statement (1) says that $x^3 > y$, we can deduce that $x > y$.

Therefore, for all possible values of x , we can say that $x > y$. Together the statements are sufficient. Answer (C).

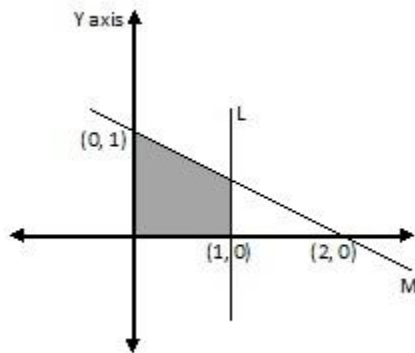
It is important to understand these relations. This concept is very useful, especially for GMAT Algebra!

37.Let's Talk about Points & Lines

A topic that has been steadily gaining ground in GMAT is co-ordinate geometry. First of all, I have to admit that I am not a fan of Geometry. Just something about learning the theorems and applying those to get the unknown angle/side makes me uncomfortable. It is easy to miss the big picture in some questions. That said, I adore co-ordinate geometry. I know that the moment I draw the diagram, the answer would be right there in front of my eyes. So let's start a discussion on co-ordinate geometry this week.

Usually, GMAT deals with two dimensional figures in the XY plane. Many questions are based on points and intersecting lines. The general form of the equation of a line is $ax + by = c$. If you are not sure of how to draw a line given its equation, check out [this post](#). Using an example, let's see how this can be helpful.

Question: In the rectangular coordinate given in the image, the shaded region is bounded by straight lines. Which of the following is NOT an equation of one of the boundary lines?



- (A) $x = 0$
- (B) $y = 0$
- (C) $x = 1$
- (D) $x - y = 0$
- (E) $x + 2y = 2$

Solution:

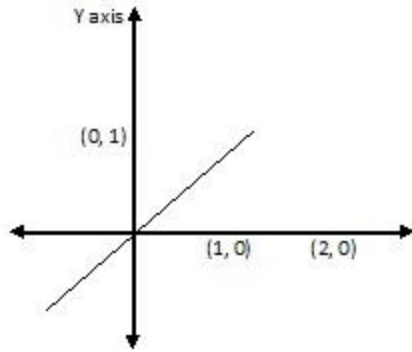
The equation of the Y axis is $x = 0$ so option (A) is not the answer.

Quarter Wit_Quarter Wisdom- Part-1

The equation of the X axis is $y = 0$ so option (B) is not the answer.

The equation of the line L is $x = 1$ so option (C) is not the answer.

This is how you represent $x - y = 0$ (which can also be written as $y = x$)



You can see that it is not one of the boundary lines. Hence, answer is (D). This is how knowing how to draw the line given the equation is useful.

You can also find questions related to the relation between two lines i.e. whether the lines are parallel or intersecting or perpendicular. Let's find out what these relations are.

Say, the equation of 2 lines is:

$$ax + by + c = 0; \text{Slope} = -a/b; \text{y intercept} = -c/b$$

and

$$mx + ny + p = 0; \text{Slope} = -m/n; \text{y intercept} = -p/n$$

If two lines intersect in a single point, their slopes will be different i.e. $-a/b \neq -m/n$

We can re-write this as $a/m \neq b/n$

If two lines are parallel, their slopes will be the same i.e. $-a/b = -m/n$

We can re-write this as $a/m = b/n$

What happens if the given two equations are of the same line? Then their slopes and their y intercepts will be the same i.e. $-a/b = -m/n$ and $-c/b = -p/n$

We can re-write this as $a/m = b/n$ and $c/p = b/n$ giving us $a/m = b/n = c/p$

Quarter Wit_Quarter Wisdom- Part-1

Hence, if you want to find two parallel lines that are distinct, ensure that their y intercept is not the same i.e. $-c/b \neq -p/n$

The relation becomes $a/m = b/n \neq c/p$

If two lines are perpendicular, the product of their slopes will be -1 i.e. $-a/b * -m/n = -1$

We can re-write this as $am = -bn$

Let me recap the relations for you:

1. A single point of intersection between two lines: $a/m \neq b/n$
2. Distinct parallel lines: $a/m = b/n \neq c/p$
3. The same line: $a/m = b/n = c/p$
4. Perpendicular lines: $am = -bn$

Take your time to be comfortable with these relations. Next week, we will take up some questions related to this concept.

38. Questions on Points & Lines

This week, we will take up some questions on co-ordinate geometry. Let me re-cap the relations we discussed in the [last post](#).

Say, the equations of 2 lines are:

$$ax + by + c = 0$$

and

$$mx + ny + p = 0$$

1. A single point of intersection between two lines: $a/m \neq b/n$
2. Distinct parallel lines: $a/m = b/n \neq c/p$
3. The same line: $a/m = b/n = c/p$
4. Perpendicular lines: $am = -bn$

Question 1: A given line L has an equation $3x+4y=5$. Which of the following is the equation of line which does not intersect the above line?

- (A) $4x + 3y = 5$
- (B) $3x + 4y = 10$
- (C) $3x + 5y = 5$
- (D) $3x + 5y = 3$
- (E) $3x - 4y = 5$

Solution: A line that does not intersect with L, is a distinct line parallel to L. The relation of coefficients between distinct parallel lines is $a/m = b/n \neq c/p$

Quarter Wit_Quarter Wisdom- Part-1

Equation of L is $3x + 4y - 5 = 0$.

$$a = 3$$

$$b = 4$$

$$c = -5$$

For option (B), $m = 3$, $n = 4$ and $p = -10$

We see that $a/m (3/3) = b/n (4/4) \neq c/p (-5/-10)$

Therefore, answer is option (B).

Question 2: What is the shortest distance between the following 2 lines: $x + y = 3$ and $2x + 2y = 8$?

- (A) 0
- (B) $1/4$
- (C) $1/2$
- (D) $\sqrt{2}/2$
- (E) $\sqrt{2}/4$

Solution:

Any two lines in the xy plane will be either parallel or intersecting. If the lines intersect, the shortest distance between them will be 0.

The two given lines are:

$$x + y = 3 \text{ (shown by the red line)}$$

$$2x + 2y = 8 \text{ which is same as } x + y = 4 \text{ (shown by the blue line)}$$

We notice that $a/m (1/1) = b/n (1/1) \neq c/p (3/4)$. Hence, the lines are parallel.

They intersect the x axis at $x = 3$ and $x = 4$ and the y axis at $y = 3$ and $y = 4$ (as shown in the figure)

Now there are many ways of getting the distance between them. The first method I will discuss is using a formula. The second method will use the properties of right triangles.

Would I advise you to learn up the formula? No. GMAT requires you to know only the very basic formulas. This is certainly not one of them. Remember it only if you have already come across it sometime during the course of your study and seeing it here is enough for you to recall it during the exam (if need be). If you are seeing this formula for the first time, don't worry about adding it to your list. There will always be other, more intuitive ways of getting to the answer.

First Method: Using the formula

Quarter Wit_Quarter Wisdom- Part-1

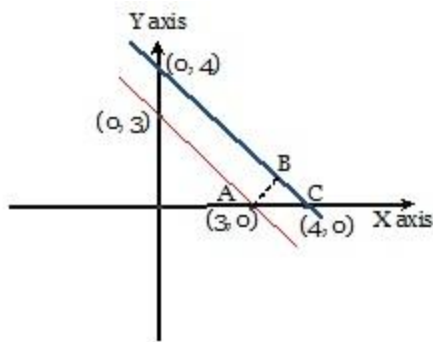
If the equations of two parallel lines are: $y = mx + b$ and $y = mx + c$ (note that they have the same slope, m , but different y intercepts, b and c)

Distance between them = $|b-c|/\sqrt{m^2 + 1}$

Here the parallel lines are: $y = -x + 3$ and $y = -x + 4$

Distance between them = $|4-3|/\sqrt{(-1)^2 + 1} = 1/\sqrt{2} = \sqrt{2}/2$

Second Method: Using right triangles



Use the little triangle ABC. Co-ordinates of A are (3, 0) and of C are (4, 0).

When you draw the red line, you notice that its x and y intercepts are the same.

i.e. $x + y = 3$ intersects x axis at 3 and y axis also at 3. So it forms an isosceles triangle.

Similarly, $x + y = 4$ intersects x axis at 4 and y axis also at 4. It also forms an isosceles triangle so angle BCA is 45 degrees.

AB is dropped perpendicular to the blue line. This is the distance between the two parallel lines. Since angle ABC is 90 degrees, angle BAC will also be 45 degrees (to make the sum 180). So $AB = AC$.

In an isosceles right triangle, the ratio of the sides is 1:1: $\sqrt{2}$ where $\sqrt{2}$ is the hypotenuse. Since we know that the hypotenuse is actually 1 ($= 4 - 3$), the measure of equal sides (AB and BC) will be $1/\sqrt{2}$ each. Multiply and divide this by $\sqrt{2}$ to get $\sqrt{2}/2$.

Hence, the distance between the two lines, $AB = \sqrt{2}/2$.

Answer (D).

Hope the application of the concept discussed is clear. We will continue working on co-ordinate geometry next week. Till then, keep practicing!

39. Questions on Intercepts & Vertices

Continuing on our quest to master coordinate geometry, let's look at a couple of Data Sufficiency questions today. The first question uses a great little concept. The second question has a very important takeaway that we all know but hardly ever implement.

Quarter Wit_Quarter Wisdom- Part-1

Question 1: In the xy-coordinate plane, line l and line k intersect at the point (5, 4). Is the product of their slopes negative?

- 1) The product of the x-intercepts of lines l and k is positive.
- 2) The product of the y-intercepts of lines l and k is negative.

Solution:

Many people start drawing lines to figure out the answer here. Most are able to get the correct answer but they just take far too much time trying out various cases – making l with x intercept positive, k with x intercept positive, then l with x intercept negative etc. What we forget here is something we know intuitively but never use:

Slope of a line = $-(y \text{ intercept})/(x \text{ intercept})$

If you are wondering why it is so, think what 'intercept' represents...

The point of x intercept is (X, 0) (where y co-ordinate is 0). We say the x intercept is 'X'.

The point of y intercept is (0, Y) (where x co-ordinate is 0). We say the y intercept is 'Y'.

So the line passes through these two points: (X, 0) and (0, Y)

If we know two points through which the line passes, we know we can find its slope. If the two points are (x1, y1) and (x2, y2), we know the slope = $(y2 - y1)/(x2 - x1)$.

Here the two points that we have are (X, 0) and (0, Y)

So slope = $(Y - 0)/(0 - X) = -Y/X$

If you remember, Y was the y intercept and X was the x intercept. Therefore, slope is given by $-(y \text{ intercept})/(x \text{ intercept})$ in terms of intercepts.

Let's use this discovery to quickly solve the question now.

Let us say that the x intercept of line l is xl and y intercept of line l is yl. Also, let us say that x intercept of line k is xk and y intercept of line k is yk

Slope of line l = $-yl/xl$

Slope of line k = $-yk/xk$

Product of the slopes of lines l and k = $(yl*yk)/(xl*xk)$

Statement 1: The product of the x-intercepts of lines l and k is positive.

We are given that $xl*xk$ is positive. But we have no information about $yl*yk$. Hence, this statement alone is not sufficient.

Statement 2: The product of the y-intercepts of lines l and k is negative.

We are given that $yl*yk$ is negative. But we have no information about $xl*xk$. Hence, this statement alone is not sufficient.

Using both together, we know that $xl*xk$ is positive and $yl*yk$ is negative. Hence $(yl*yk)/(xl*xk)$ must be negative. Sufficient.

Answer (C).

Concept to Remember: Slope of a line is given by $(-y \text{ intercept}/x \text{ intercept})$ in terms of intercepts.

Let's go on the next question now.

Question 2: On the xy-coordinate plane, a triangular region is bounded by the lines $y = 3$, $x = -6$, and $y = cx + d$. One vertex of this region is (-6,0). What is the perimeter of this region?

Statement 1: $d = 3$

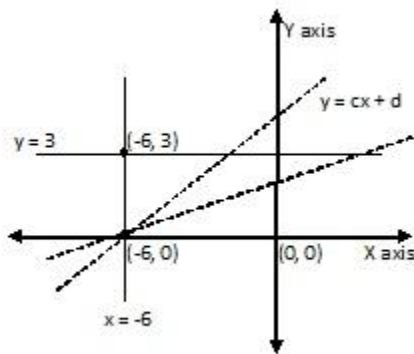
Quarter Wit_Quarter Wisdom- Part-1

Statement 2: $c = 1/2$

Solution: We all know that a Data Sufficiency question is just that – a sufficiency question. You don't actually need to solve it (with a few exceptions). All you need to do is find out whether the data is sufficient to solve it. If you forget this critical point during the exam, it will be many minutes before you finally mark (D) as your answer for this question and move on. But if you do remember this point then it will take you less than a minute to correctly reach the same answer. If you don't believe me, go ahead and solve it. Clock the time you take. Then think how you could have figured out the answer without actually solving it.

Let me give you my train of thought:

Let's first see what the triangle looks like.



The dotted lines show the third side. Since we do not know the values of c and d , we cannot plot the line accurately. It could look like either of the two dotted lines shown or something similar. The point $(-6, 3)$ will be a vertex of the triangle since it is the point of intersection of $x = -6$ and $y = 3$. Since $(-6, 0)$ is given as a vertex too, the line $y = cx + d$ must pass through it.

To get the perimeter of the triangle, you need to know the length of the sides. To know the length of the sides, you need the co-ordinates of the vertices. (If you know the two vertices (x_1, y_1) and (x_2, y_2) , the length of the side is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$).

To know the co-ordinates of the vertices, you need to know the equations of all the lines. (When you know the equation of two lines, you can find their point of intersection by solving them simultaneously.)

You know the equation of two lines: $y = 3$ and $x = -6$. But you do not know the equation of the third line, $y = cx + d$. All that is missing to get the perimeter of the triangle is the equation of this line. c is the slope of this line and d is the y intercept. We see from the diagram that this line passes through $(-6, 0)$ so we already have one point through which it passes. Let's look at the statements now.

Statement 1: $d = 3$

This statement tells us that the y intercept is 3. This means that this line passes through $(0, 3)$. So now we know 2 points through which this line passes $(-6, 0)$ and $(0, 3)$. Hence we can find the equation of the line. This statement alone is sufficient to get the perimeter of the triangle.

The equation of a line when two points, (x_1, y_1) and (x_2, y_2) , through which it passes are given, is $y - y_1 = \frac{(y_2 - y_1)(x - x_1)}{(x_2 - x_1)}$

Statement 2: $c = 1/2$

Quarter Wit_Quarter Wisdom- Part-1

This statement tells us that the slope of the line is $1/2$. Now we know a point through which this line passes $(-6, 0)$ and its slope. The slope and a point are sufficient to find the equation of the line. This statement alone is sufficient to get the perimeter of the triangle.

The equation of a line when a point, (x_1, y_1) , through which it passes and the slope, m , are given, is $y - y_1 = m(x - x_1)$

Since both statements are individually sufficient to get the perimeter of the triangle, the answer is (D).

Takeaway: Data Sufficiency questions needn't be solved. You just need to know whether the given data is sufficient to solve!

40. Use Variations to Succeed

The key to doing well on GMAT Quant is to pick your standard questions, understand them really well and then try out variations of these questions. A small change in the question might force you to rethink your entire approach. The more you experiment, the more interesting your GMAT preparation will get and not to forget, the stronger your Quant will be.

Let me show you what I mean with the help of an example. Since we have been doing co-ordinate geometry for the past few weeks, let's look at an interesting Official Guide question on coordinate geometry. Thereafter, we will try and figure out some variations of the same. We will focus more on the variations and how to think dynamically to arrive at solutions in those cases.

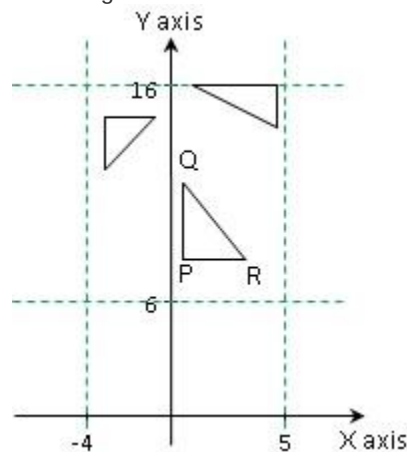
Official Guide Question: Right triangle PQR is to be constructed in the xy -plane so that the right angle is at P and PR is parallel to the x -axis. The x and y coordinates of P, Q and R are to be integers that satisfy the inequalities $-4 \leq x \leq 5$ and $6 \leq y \leq 16$. How many different triangles with these properties can be constructed?

- (A) 110
- (B) 1100
- (C) 9900
- (D) 10000
- (E) 12100

Solution:

The triangle should be right at P and PR should be parallel to x axis. Also, the vertices P, Q and R should lie within or on the boundary of the green dotted rectangle shown in the diagram.

The triangle can take various orientations as shown.



Consider the green dotted rectangle. The x coordinate varies from -4 to 5 and the y coordinate varies from 6 to 16 .

PR has to be parallel to x axis. Since it is a right angled triangle, PQ will be parallel to y axis.

Quarter Wit_Quarter Wisdom- Part-1

Now let's first fix the vertex P. In how many ways can you choose the coordinates of P? The x coordinate of P can vary from -4 to 5 i.e. it can take 10 values. The y coordinate of P can vary from 6 to 16 i.e. it can take 11 values. In all, the coordinates of P can take $10 \times 11 = 110$ values.

Once vertex P is fixed, the x coordinate of Q will be the same as the x coordinate of P (since PQ is parallel to y axis) and the y coordinate of R will be the same as the y coordinate of P (since PR is parallel to x axis).

The y coordinate of Q can be chosen in 10 ways. (Out of the 11 values of y coordinate, one is occupied by P so 10 are leftover.)

The x coordinate of R can be chosen in 9 ways. (Out of 10 values of x coordinate, one is occupied by P so 9 are leftover.)

Total number of ways of making the triangle = $110 \times 10 \times 9 = 9900$

Note that we have counted all possible triangles of all orientations in this case.

Answer: C

Hope you understand the method discussed. Here we were focusing on right triangles only. That made our task a little easier.

What if we remove this condition? What if we had to find the total number of triangles? Let's try a variation without the right triangle condition.

Variation 1: How many triangles with positive area can be drawn on the coordinate plane such that the vertices have integer coordinates (x,y) satisfying $2 \leq x \leq 4$ and $5 \leq y \leq 7$?

(A) 72

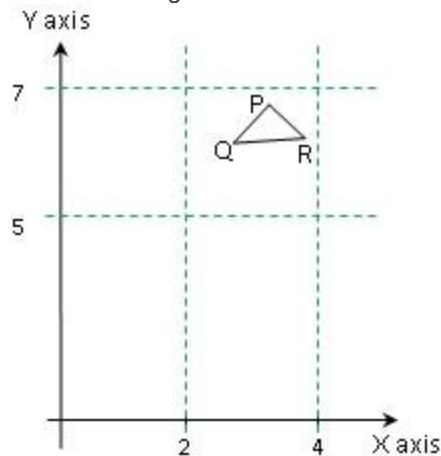
(B) 76

(C) 78

(D) 80

(E) 84

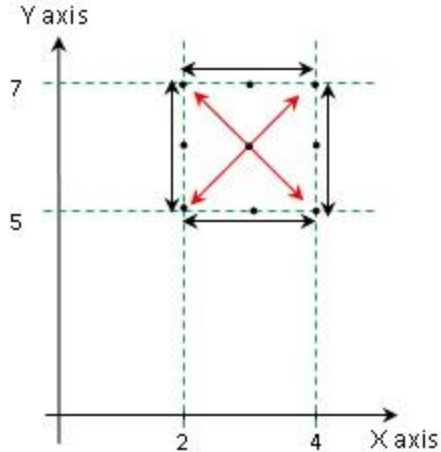
Let's draw the figure first.



With the given conditions, $2 \leq x \leq 4$ and $5 \leq y \leq 7$, the triangle should lie in the green dotted rectangle, as shown. For a vertex, the x coordinate can be chosen in 3 ways (2, 3 or 4) and the y coordinate can be chosen in 3 ways (5, 6 or 7). Hence we can choose a vertex in $3 \times 3 = 9$ ways (e.g. (2, 5), (2, 6), (2, 7), (3, 5) etc). We need 3 vertices to form a triangle. The first vertex can be chosen in 9 ways, the second vertex can be chosen in 8 ways and the third vertex can be chosen in 7 ways. Hence, there are a total of $9 \times 8 \times 7$ ways of choosing the three vertices. But since we have ordered the vertices as first, second and third in this case, we need to divide this number by 3! (If this is unclear, don't worry. I will take up Permutations and Combinations next where we will discuss these concepts in detail.)

But these $9 \times 8 \times 7 / 3!$ cases also include those where all three vertices selected are collinear.

Quarter Wit_Quarter Wisdom- Part-1



3 sets of horizontally collinear points: (shown by 2 horizontal black arrows)

(2, 5), (3, 5) and (4, 5)

(2, 6), (3, 6) and (4, 6)

(2, 7), (3, 7) and (4, 7)

3 sets of vertically collinear points: (shown by 2 vertical black arrows)

(2, 5), (2, 6), (2, 7)

(3, 5), (3, 6), (3, 7)

(4, 5), (4, 6), (4, 7)

2 sets of diagonally collinear points: (shown by 2 red arrows)

(1, 1), (2, 2), (3, 3)

(1, 3), (2, 2), (3, 1)

We need to subtract these from $9 \times 8 \times 7 / 3!$ because collinear points do not make a triangle.

All other sets of 3 points will make a triangle.

Total number of triangles = $84 - 8 = 76$

Answer (B)

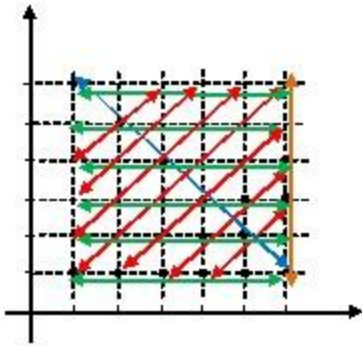
We used brute force method in this question. We counted the number of collinear points and subtracted them from the total available number of coordinates. The problem of collinear points did not occur in the previous question because we were considering just right triangles there. What happens in case the acceptable region i.e. the dotted green rectangle is much larger? How will we calculate all the collinear points in that case? Let's see.

Variation 2: How many triangles with positive area can be drawn on the coordinate plane such that the vertices have integer coordinates (x, y) satisfying $1 \leq x \leq 6$ and $1 \leq y \leq 6$?

In this question, there are a total of 36 co-ordinates to choose from. We need to make triangles so we need to select a triplet of co-ordinates out of these 36 which can be done in $36 \times 35 \times 34 / 3!$ ways (same logic as above). Out of these, we need to get rid of those triplets where the points are collinear. How many such triplets are there? There are many sets which are horizontally, vertically and diagonally collinear. How do we count all these?

Look at the diagram given below:

Quarter Wit_Quarter Wisdom- Part-1



There are six horizontally collinear points for each y co-ordinate shown by the green lines. From each of these 6 horizontal collinear points, you can select any 3 and they will not give you a triangle. You can select 3 points out of 6 in $6 \cdot 5 \cdot 4 / 3!$ ways (or $6C3$). There are 6 such groups of 6 collinear points so we can select 3 collinear points (horizontally) in $6 \cdot 6 \cdot 5 \cdot 4 / 3!$ ways = 120 ways.

Same is the case with vertically collinear points. There are six vertically collinear points for each x co-ordinate shown by the single orange line. From each of these 6 vertically collinear points, you can select any 3 and they will not give you a triangle. You can select 3 points out of 6 in $6 \cdot 5 \cdot 4 / 3!$ ways (or $6C3$). There are 6 such groups of 6 collinear points so we can select 3 collinear points (vertically) in $6 \cdot 6 \cdot 5 \cdot 4 / 3!$ ways = 120 ways.

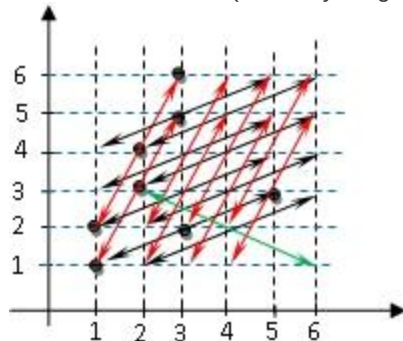
Now let's look at the diagonally collinear points. The red arrows show the diagonally collinear points. You can select 3 points from the 3 collinear points in 1 way (shown by the smallest red arrow). You can select 3 points from a set of 4 collinear points in $4 \cdot 3 \cdot 2 / 3! = 4$ ways. You can select 3 points from a set of 5 collinear points in $5 \cdot 4 \cdot 3 / 3! = 10$ ways. You can select 3 points from a set of 6 collinear points in $6 \cdot 5 \cdot 4 / 3! = 20$ ways. Now again you have 5 collinear points so you can select 3 out of them in 10 ways. Now you have 4 collinear points so you can select 3 out of them in 4 ways and from the remaining 3 collinear points, you can select 3 in 1 way.

In all, you can select 3 diagonally collinear points in $1+4+10+20+10+4+1 = 50$ ways in this direction.

Now consider the opposite alignment shown by the blue arrow. Here again, you can select 3 collinear points in 50 ways (just like above)

Now consider collinear points lying on lines which have a slope of 2 or $1/2$ i.e. for a change of 1 unit of one co-ordinate, the other co-ordinate changes by 2.

Look at the red arrows below. When x co-ordinate changes by 1, the y co-ordinate changes by 2. The points (1, 1), (2, 3) and (3, 5) lie on same line i.e. they are collinear. You have 8 red arrows and 8 black arrows. Similarly, you will have another set of 16 from the other end (shown by the green arrow)



Quarter Wit_Quarter Wisdom- Part-1

Total sets of 3 collinear points = $120 + 120 + 50 + 50 + 32 = 372$

Therefore, total number of triangles we can make in this case = $36 \times 35 \times 34 / 3! = 372$

Each of the two variations required a twist to your strategy. The first variation was simpler since you just had to manually calculate the number of collinear points. The second one needed a little more thinking. I hope this example will encourage to try out some variations of the next interesting Quant question you come across.

41.Magic or Math

We say goodbye to GMAT coordinate geometry for a while now and start today's post with some magic tricks:

Trick 1

Step 1: Pick any two consecutive integers (Don't tell me what they are!).

Step 2: Multiply them.

Whatever your numbers, the product is an even number! If you are wondering how I knew that, you desperately need to read this post. If you are shaking your head in disappointment, wait, I have more.

Note: 0 is even. So are -2, -4, -6 etc

Trick 2

Step 1: Pick any three consecutive integers.

Step 2: Multiply them.

Whatever your numbers, the product is a multiple of 6. Now, if you are surprised, great, go ahead and read this post. If you are still bewildered that why the heck am I calling 'simple Math' 'magic tricks', wait, give me one last chance.

Trick 3

Step 1: Pick any four consecutive integers.

Step 2: Multiply them.

Whatever your numbers, the product is definitely a multiple of 24. I hope I have caught your interest now. If you are still not surprised, tell me the greatest number which is definitely a factor of the product of any five consecutive integers. If you do get an answer, scroll down to the bottom of the post to check it. If you get the correct answer, you don't need to read this post. Hopefully, I will see you next week with something more challenging. If not, then stick around.

So how did I know the numbers which were definitely factors of consecutive integers? Let me explain you the simple Math involved.

Theory:

Quarter Wit_Quarter Wisdom- Part-1

Pick any two consecutive numbers. e.g. (5, 6) or (101, 102) or (999, 1000) etc ..

Do you agree that one of them, and only one of them will be even? Every alternate number has 2 as a factor so no matter which two consecutive numbers you pick, one of them will definitely have 2 as a factor and the other will not.

Notice a few things about integers:

...-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16...

- Every number is a multiple of 1
- Every second number is a multiple of 2
- Every third number is a multiple of 3
- Every fourth number is a multiple of 4 and so on...

If we pick any 2 consecutive integers, one and only one of them will be a multiple of 2: e.g. pick 4, 5 (4 is a multiple of 2) or pick 11, 12 (12 is a multiple of 2) etc.

If we pick any 3 consecutive integers, at least one of them will be a multiple of 2 e.g. pick 3, 4 and 5 (4 is a multiple of 2) or pick 6, 7 and 8 (6 and 8 are multiples of 2) etc. Of the 3 numbers, exactly one will be a multiple of 3 e.g. pick 3, 4 and 5 (3 is a multiple of 3) or pick 6, 7 and 8 (6 is a multiple of 3) etc. Hence the product of the 3 consecutive integers will be a multiple of 2 and 3 and therefore, of 6.

If we pick any 4 consecutive integers, two of them will be multiples of 2 e.g. pick 3, 4, 5 and 6 (4 and 6 are multiples of 2) or pick 6, 7, 8 and 9 (6 and 8 are multiples of 2) etc. Of the 4 numbers, at least one will be a multiple of 3 e.g. pick 3, 4, 5 and 6 (3 and 6 are multiples of 3) or pick 5, 6, 7 and 8 (6 is a multiple of 3). Also, exactly one number will be a multiple of 4 e.g. pick 3, 4, 5 and 6 (4 is a multiple of 4) or pick 5, 6, 7 and 8 (8 is a multiple of 4) etc. Hence the product will be a multiple of at least 4, 3 and 2 (from the even integer which is not a multiple of 4) and therefore, of 24.

Hope the concept is clear to you. Let's look at a question now.

Question: Given that n is any integer such that $(n-1)*n*(n+1)$ is divisible by 24, which of the following must be true?

1. Either n is divisible by 8 or $(n+1)$ is divisible by 4
2. Either n or $(n^2 - 1)$ is divisible by 3
3. n is not divisible by 16

Solution:

$(n-1)$, n and $(n+1)$ are consecutive integers. Either $(n-1)$ and $(n+1)$ are even or n is even. We know that n has 8 and 3 as factors. If n is even, it will be a multiple of 8 since $(n-1)$ and $(n+1)$ will be odd. If $(n-1)$ and $(n+1)$ are even, at least one of them will be a multiple of 4 (to get 8 as the factor of their product). Either $(n-1)$ or $(n+1)$ could be the integer divisible by 4. Hence statement I is not necessarily true.

Since the product is divisible by 24, one of $(n-1)$, n and $(n+1)$ must be divisible by 3. Either n or $(n-1)(n+1) = (n^2 - 1)$ must be divisible by 3. Hence statement II must be true.

Quarter Wit_Quarter Wisdom- Part-1

If the three consecutive integers are 15, 16 and 17, n could be divisible by 16. All we know is that the product is divisible by 24, it could be divisible by 48 or higher numbers. So statement 3 is not necessarily true.

Answer to the 5 consecutive integers question: 120 (Shortcut – If you need 5 consecutive integers, take 1, 2, 3, 4 and 5 – their product is 120. This will be the maximum number which will definitely be a factor of any five consecutive integers.)

42.Magic or Math- II

Today we will continue from where we left our [last post](#). In the last post, we discussed that of any two consecutive integers, one and only one of them will be even. Out of 20 and 21, only 20 is even. Since 2 is a factor of 20, it will not be a factor of 21. Does that make sense? Sure. Every second number will have 2 as a factor.

On the same lines, can both the consecutive numbers have 3 as a factor? Let's take the same example – 20 and 21. 3 is a factor of 21. Can it be a factor of 20 as well? Do we even need to check? Since 21 is a multiple of 3, the previous multiple must be 3 places before (i.e., 18) and the next multiple of 3 must be 3 places ahead (i.e., 24).

What do you conclude then? Two consecutive integers can only have 1 common factor and that is 1. This means that if we pick any two consecutive integers, they will have no common factor other than 1. Say if 5 were their common factor, the numbers would be 5/10/15... apart e.g. 25 and 30. They cannot be consecutive. If 11 were their common factor, the numbers would be at least 11 apart e.g. 11 and 22. They cannot be consecutive.

Out of three consecutive integers, two could have 2 as a common factor e.g. 20, 21 and 22. Both 20 and 22 have 2 as a common factor. But can 3 be a common factor of any two numbers? No. One and only one number will be a multiple of 3.

Another way to look at this – Say we have the following consecutive integers:

$(N - 4), (N - 3), (N - 2), (N - 1), N, (N + 1), (N + 2), (N + 3), (N + 4)$

We are given that 2, 3, 5 and 7 are factors of N . What can we say about the factors of the rest of the numbers?

1. $(N + 1)$ and $(N - 1)$ both will NOT have any of 2, 3, 5 and 7 as factors. They are consecutive with N . If N is a multiple of 2, 3, 5 and 7, the next multiples of these numbers will be farther away.
2. $(N + 2)$ and $(N - 2)$ both will have 2 as a factor. They are 2 steps away from N . Since N is a multiple of 2, they will be multiples of 2 too. $(N + 2)$ and $(N - 2)$ both will NOT have 3, 5 and 7 as factors. $(N + 2)$ and $(N - 2)$ are only two steps away from N . The next multiples of 3, 5 and 7 will be farther away.
3. $(N + 3)$ and $(N - 3)$ will have 3 as a factor. They both are 3 steps away from N . Since 3 is a factor of N , it will also be a factor of these two numbers. They will not have 2, 5 and 7 as factors.
4. $(N + 4)$ and $(N - 4)$ will have 2 as a factor. They are 4 steps away from N . Since N is a multiple of 2, they will be multiples of 2 too. But, they will NOT have 3, 5 and 7 as factors.

The diagram given below will help you visualize this concept.

Quarter Wit_Quarter Wisdom- Part-1

N-5	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4	N+5
205	206	207	208	209	210	211	212	213	214	215
↓	↓	↓	↓		↓		↓	↓	↓	↓
	2		2		2		2		2	
		3			3			3		
5					5					5
					7					

Now think: If you pick any two consecutive integers, can they both have 4 as a factor? or 7 as a factor? or 99 as a factor? No! Once you get one multiple of 99, you will not get another one in the next 98 numbers. The next multiple will appear when you add 99 to this multiple. For example, say you pick 99. Can 100, 101, 102... be multiples of 99? No. The next multiple of 99 will be 198. Therefore, numbers from 100 to 197 will not be multiples of 99.

We can say that consecutive numbers will not have any common factor other than 1. (1 is a factor of every number.)

Let's look at how knowing this property can be useful.

Question: For every positive even integer n , the function $f(n)$ is defined to be the product of all the even integers from 2 to n , inclusive. If p is the smallest prime factor of $f(100) + 1$, then p is

- (A) between 2 and 20
- (B) between 10 and 20
- (C) between 20 and 30
- (D) between 30 and 40
- (E) greater than 40

Solution:

First of all, the question sounds much more complicated than it actually is. Just put some values for n and try and figure out what the function looks like.

$$f(2) = 2$$

$$f(4) = 2 \cdot 4$$

$$f(6) = 2 \cdot 4 \cdot 6$$

and so on...

$$f(100) = 2 \cdot 4 \cdot 6 \cdot \dots \cdot 98 \cdot 99 \cdot 100 = (2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot \dots \cdot (2 \cdot 48) \cdot (2 \cdot 49) \cdot (2 \cdot 50)$$

$$f(100) = (2^{50}) \cdot 1 \cdot 2 \cdot 3 \cdot \dots \cdot 48 \cdot 49 \cdot 50$$

Quarter Wit_Quarter Wisdom- Part-1

Can I say that all numbers from 2 to 50 are definitely factors of $f(100)$? Sure. We can see above that they are. Then what can I say about the factors of $f(100) + 1$? Since $f(100)$ and $f(100) + 1$ are consecutive integers, can I say that they share only one common factor and that is 1? Yes, I can. We just saw this concept above. This means that $f(100) + 1$ will not have any factor lying between 2 and 50, inclusive because each of these numbers is a factor of $f(100)$. So if p is a prime factor of $f(100) + 1$, can we say that it must be greater than 50? Yes, we can. We know that p cannot be 1 since p is a prime factor. The next factor of $f(100) + 1$ must be greater than 50. Since it is greater than 50, it will definitely be greater than 40 too.

Answer (E)

I think you will agree that the solution is much simpler than what you would have first expected. This is an important number property and could be useful in a range of situations. Make sure you understand it well, and as always, keep practicing!

43.The Dreaded Combinatorics

After much deliberation, I have decided to start with Combinatorics and Probability this week. Why after much deliberation? Because I know that once I get *into it*, it will be many weeks before I get *out of it*. Anyway, I think it's time we touch upon some important concepts of this vast topic. Mind you, I will stick to the GMAT-relevant sections so if you have any out-of-GMAT-scope intellectual questions, send them to me on my mail id (kbansal@veritasprep.com) and we can discuss those on the side.

The first thing I want to discuss is something we call "Basic Counting Principle" because it is useful in almost all 600-700 level questions of Combinatorics (Note here that I will avoid using the terms "Permutation" and "Combination" and the formulas associated with them since they are not necessary and make people uncomfortable). Also, many of the 700+ level questions use basic counting principle as the starting point so it's not possible to start a discussion on combinatorics without discussing this principle first. Let's try to understand it using an example.

Example 1: There are 3 boys and 2 girls. We want to select a pair of one boy and one girl for a dance. In how many ways can we do it?

Solution: Let's discuss the solution in detail. This is a very basic and very important concept.

Say the 3 boys are B1, B2 and B3. Say the 2 girls are G1 and G2.

In how many ways can you make a pair?

B1 – G1

B1 – G2

B2 – G1

B2 – G2

B3 – G1

Quarter Wit_Quarter Wisdom- Part-1

B3 – G2

A total of 6 ways. We see that we can select a boy in 3 ways (since there are 3 boys) and we can select a girl in 2 ways (since there are 2 girls). So we can make a pair in $3 \times 2 = 6$ ways.

The basic counting principle deals with problems having 'distinct spots' and 'available contenders'. Here we have 1 spot for a boy and 1 spot for a girl i.e. 2 distinct spots. There are 3 contenders for the empty 'boy spot' and 2 contenders for the empty 'girl spot'. These spots can be filled in $3 \times 2 = 6$ ways. The word 'distinct' is important here as we will see in the next few weeks.

Also notice here that it is not $3 + 2 = 5$ ways. This is so because we have to choose a boy AND a girl simultaneously. For every boy, we could choose a girl in 2 ways and there are 3 boys so we can choose a pair in 3×2 ways. If we had to choose one boy OR one girl (i.e. just one person), we could have done it in $3 + 2 = 5$ ways because there are 5 people and we have to choose one of them. The distinction between 'AND' and 'OR' is quite important since it defines whether you will multiply or add.

Let's look at some more examples of basic counting principle.

Example 2: A restaurant serves 6 varieties of appetizers, 10 different entrees and 4 different desserts. In how many ways can one make a meal if one chooses one appetizer, one entree and one dessert?

Solution: In how many ways can you choose an appetizer? 6 ways

In how many ways can you choose your entree? 10 ways

In how many ways can you choose your dessert? 4 ways

In how many different ways can you make your meal? To make your meal, you need one appetizer AND one entree AND one dessert. There are 3 distinct spots and you have to fill each one of them. You can do it in $6 \times 10 \times 4 = 240$ ways.

Example 3: In how many ways can you make a five letter password using the first ten letters of the English alphabet? (You can use only capital letters.)

Solution: In how many ways can you choose the first letter of the password? 10 ways. You can put in any letter from A to J.

What about the second letter? Again, you can choose it in 10 ways. The question doesn't say that you cannot repeat a letter once you use it.

Similarly, each digit can be chosen in 10 ways.

Since you have to choose the first letter AND the second letter AND the third letter etc, the total number of ways of selecting a five letter password is $10 \times 10 \times 10 \times 10 \times 10 = 10^5$ ways. You have 5 distinct spots and you can fill each one of them in 10 ways.

Quarter Wit_Quarter Wisdom- Part-1

You can make the 5 letter password in 10^5 ways.

Example 4: Five friends go to watch a movie. They are supposed to occupy seat numbers 51 to 55. In how many different ways can they do it?

Solution: Let's look at the first seat i.e. seat number 51. In how many ways can you make someone sit there? Of course 5 ways since you have 5 people. Say, you make one of them sit on seat number 51. Now in how many ways can you make someone sit on seat number 52? Remember that you have only 4 contenders left now since one of them is already sitting on seat number 51. Therefore, you can make someone sit there in 4 ways only. Next, for seat number 53, you only have 3 contenders left. For seat number 54, you have only 2 contenders left and then for seat number 55, only the last person is remaining i.e. just one option.

The total number of ways of arranging 5 people on 5 distinct seats is $5*4*3*2*1 = 120$

These are some of the most basic examples of basic counting principle. Below, I am giving more questions which are just twists on these questions. Try them and get back if you have any doubts. We will take some higher level concepts from next week on.

Question 1: A restaurant serves 6 varieties of appetizers, 10 different entrees and 4 different desserts. In how many ways can one make a meal if one chooses an appetizer, at least one and at most two different entrees and one dessert?

Question 2: In how many ways can you make a five digit password using the first ten letters of the English alphabet if each letter can be used at most once? (You can use only capital letters.)

Question 3: In how many ways can BRIAN make a five digit password using the first ten letters of the English alphabet if each letter can be used at most once and one is not allowed to use any letter which appears in one's name? (You can use only capital letters.)

Question 4: Six friends go to watch a movie. They are supposed to occupy seat numbers 51 to 56. However one of them falls sick and returns home. In how many different ways can the 5 people sit?

Question 5: Eight friends go to watch a movie but only 5 tickets were available. In how many different ways can 5 people sit and watch the movie?

44.The Circular Arrangements

Let's start this post with a question: In how many different ways (relative to each other) can 5 friends sit around a round table if all the seats are identical?

I guess that most of you will be able to answer it $\rightarrow 4! = 24$ ways

After all, you know the formula of circular arrangement which is $(n-1)!$ But, many of you probably do not understand exactly why the formula is $(n-1)!$ Today's post is focused on explaining the concept of circular arrangement. If you are wondering why

Quarter Wit_Quarter Wisdom- Part-1

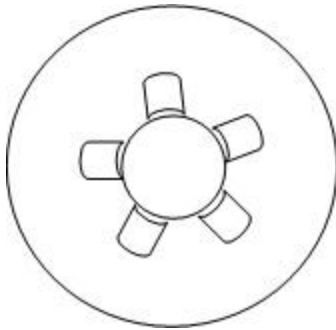
you need to know the theory behind the formula when all you need to do in a question is apply the formula and get the answer, here is why — you will be able to solve a straight forward 500-600 level question knowing just the formula but you will not get the 700+ level GMAT question correct. You need to understand the basics behind the formula so that you can apply it with modifications in more inventive situations. I will give you a couple of questions after discussing the theory and you will see what I mean. Right now, let's focus on the question posed above.

Question 6: In how many different ways (relative to each other) can 5 friends sit around a round table if all the seats are identical?

Now there are two ways to explain the formula used here. I will give both. See what makes more sense to you.

Method 1:

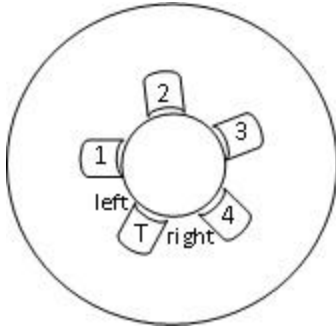
When we pose questions on circular arrangement, the different arrangements we are looking for are those in which people are sitting differently relative to each other. So ignore any other point of reference. Say there is a monochromatic circular room with a circular door right in the middle of the roof and people enter the room Mission-Impossible style. There is a circular table with 5 chairs around it all placed at equal distances. When the first person, Mr. T, drops in, which chair should he sit on? Can we say that it is immaterial which chair he sits on since all the seating spaces are exactly the same?



Since there is no one else sitting as yet, for him every seat is the same. In how many ways can he choose a seat then? In only one way (since every way in which he chooses a seat is the same). No matter which chair he sits on, the arrangement remains exactly the same.

Now when the second person, Mr. B, drops in, in how many different ways can he occupy a seat? Mr. B has four choices. Each one of the seats is different relative to where Mr. T. sits. Mr. B can choose to sit on the left of Mr. T, next to him — i.e., on seat number 1. Or he can choose to sit on the left of Mr. T but with a seat between them i.e. seat number 2. Or he can sit on the right of Mr. T, next to him i.e. seat number 4. Or he can sit on the right of Mr. T but with a seat between them i.e. seat number 3. Each one of the 4 seats are different relative to Mr. T.

Quarter Wit_Quarter Wisdom- Part-1



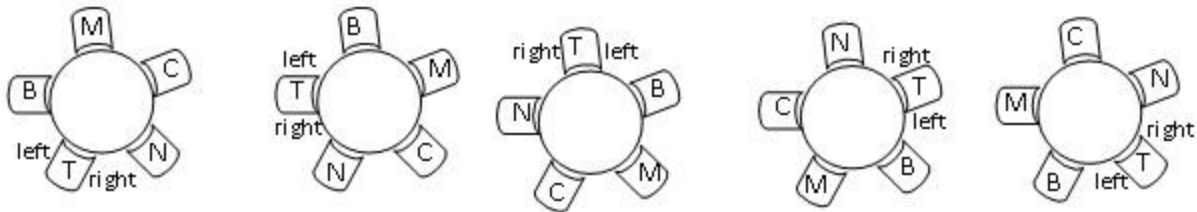
So Mr. B can choose a seat in 4 ways.

Next, when Mr. M drops in, he can choose a seat in 3 ways and so on till the last person has 1 seat left for him. The total number of arrangements then are $4 \times 3 \times 2 \times 1 = 24$ (think of your basic counting principle here). For the first person, all seats are the same so he can choose in 1 way. He creates a frame of reference and thereafter, every seat is distinct (relative to him). So the rest of the $(n-1)$ people can sit in $(n-1)$ seats in $(n-1)!$ ways (using Basic Counting Principle).

This is how we arrive at the formula $(n-1)!$

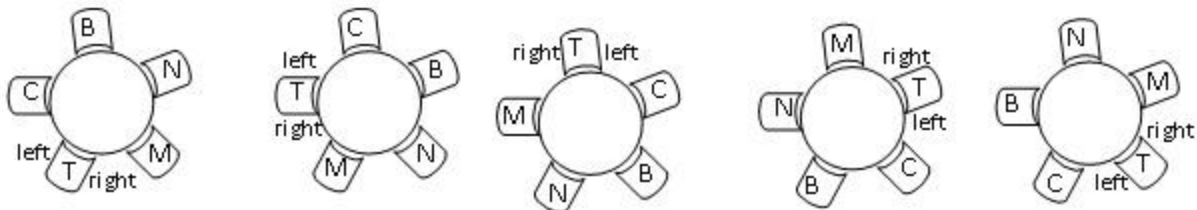
Method 2:

Let's say we have arranged the 5 people on the 5 seats. Would you say that the following 5 combinations are exactly the same (relative to people)?



Mr. B is next to Mr. T on the left and Mr. N is next to Mr. T on the right. Mr. M is to the left of Mr. B and Mr. C is to the right of Mr. N.

Similarly, these 5 arrangements are exactly the same too (but different from the arrangements above).



Mr. C is next to Mr. T on the left and Mr. M is next to Mr. T on the right. Mr. B is to the left of Mr. C and Mr. N is to the right of Mr. M.

Quarter Wit_Quarter Wisdom- Part-1

Every group of 5 arrangements is actually a single arrangement since relative to one another, people are arranged in the same way. So we divide $5!$ by 5 to count only the actual distinct arrangements.

Hence we get the formula $n!/n = (n-1)!$

I like to think in terms of method 1 since it helps me take care of a lot of variations on simple circular arrangement questions. Let's look at one of these variations.

Question 7: There are 5 people – A, B, C, D and E. They have to sit around a circular table with 5 chairs such that A can sit neither next to D nor next to E. How many such distinct arrangements are possible?

Solution: A can sit neither next to D nor next to E. So A has to sit next to B and C. Let's say we first make A sit on any one chair. In how many ways can A choose his chair? In only 1 way because all the chairs are the same for him (he is the first person being seated). Now B and C have to sit next to him. B can sit on the right of A and C can sit on the left of A OR B can sit on the left of A and C can sit on the right of A. There are two ways in which you can arrange B and C around A. Now there are 2 chairs left and two people D and E. D can choose his chair in 2 ways since the two seats are distinct (relative to A, B and C) and the last chair will be left for E.

Total number of arrangements = $1 \times 2 \times 2 \times 1 = 4$ ways

Note: In case nothing is mentioned, in a circular arrangement, two seating arrangements are considered different only when the positions of the people are different relative to each other. If it is given that the seats are distinct (say they are different colored), then the number of arrangements is $n!$ (same as in the case of linear arrangements)

Now, let me leave you with a question which is based on the concept of circular arrangement and can be easily solved if you understand the theory above. I will discuss its solution in the next post.

Question 7: In how many different ways (relative to each other) can 8 friends sit around a square table with 2 seats on each side of the table?

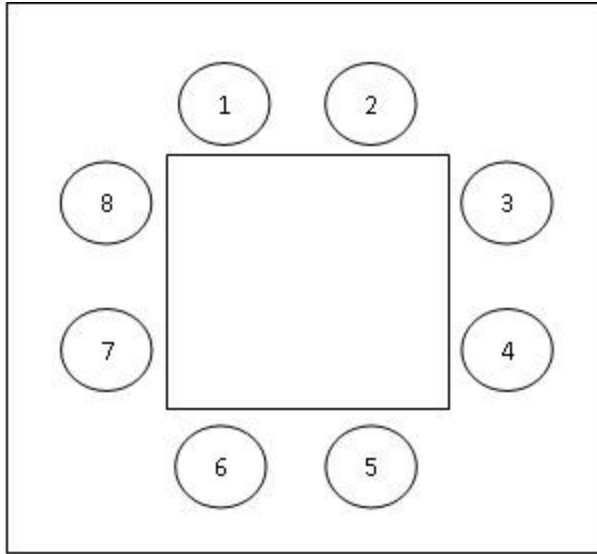
45.Linear Arrangement Constraints-I

Let me first give you the solution to the question I gave you in my last post:

Question: In how many different ways (relative to each other) can 8 friends sit around a square table with 2 seats on each side of the table?

Solution: What happens when the first friend enters the room? Are all the seats same for him?

Quarter Wit_Quarter Wisdom- Part-1



Are the seat numbers 5 and 6 the same for him? (The seats are not actually numbered. We have numbered them for easy reference.)

No, they are not! Seat no. 5 has a corner (of the table) on the right hand side and an empty chair on the left hand side while seat no. 6 has a corner on the left hand side and an empty chair on the right hand side. So then, are all the seats distinct for him? I am sure you agree that they are not. It is similar to a circle situation but not quite. Seat no. 6 and seat no. 4 are alike since they both have a corner on the left hand side and an empty chair on the right hand side.

Isn't the case the same for seat numbers 2 and 8 as well? Can I say that the seat numbers 2, 4, 6 and 8 are the same? It doesn't matter on which seat he sits out of these 4; the arrangement stays the same. Similarly, notice that seat numbers 1, 3, 5 and 7 are also the same. Therefore, there are 2 ways in which the first person can sit. He can either sit on one of 1, 3, 5 and 7 or on one of 2, 4, 6 and 8. After he sits, all the remaining 7 seats are distinct. 7 people can sit on 7 distinct seats in $7!$ ways.

Total number of arrangements = $2 \cdot 7!$

Hope it makes sense to you, especially for GMAT prep purposes. You can try any number of variations now. You can also try putting in constraints. I will focus on constraints in linear arrangements today. Perhaps, in another post, we can look at constraints in circular arrangements. In the first post on combinatorics, we learned the basic counting principle. Using that we can solve many simple questions for example:

Question 1: In how many ways can 3 people sit on 3 adjacent seats of the front row of the theatre?

We know that it is a straight forward basic counting principle application. On the leftmost seat, a person can sit in 3 ways (choose any one of the 3 people). On the middle seat, a person can sit in 2 ways (since 1 person has already been seated). There is only 1 person left for the rightmost seat so he can sit there in 1 way. Total number of arrangements = $3 \cdot 2 \cdot 1 = 3! = 6$

Now, let's try to add some constraints here. I will start with some very simple constraints and go on to some fairly

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advanced constraints.

Question 2: In how many ways can 6 people sit on 6 adjacent seats of the front row of the theatre if two of them, A and B, cannot sit together?

Solution: This is a very simple constraint question. First tell me, what if there was no constraint i.e. what is the total number of arrangements in which 6 people can sit in a row? You should know by now that it is $6!$ (using Basic Counting Principle). Now, rather than counting the number of ways in which they will not sit next to each other, we can count the number of ways in which they will sit next to each other and subtract that from the total number of arrangements. Why? Because it is easier to group them together (think that we have handcuffed them together) and treat them as a single person to get the arrangements where they will sit together.

So let's deal with a different question first: In how many ways can 6 people sit on 6 consecutive seats of the front row of the theatre if two of them, A and B, must sit together?

There are 5 individuals/groups: AB C D E F

These 5 can be arranged in a line in $5!$ ways. But the group AB itself can be arranged in 2 ways AB or BA i.e. B could be to the right of A or to the left of A.

Total number of arrangements = $2 \times 5!$ (Notice the multiplication sign here. We have to arrange the 5 individuals/groups AND A and B.)

This is the number of arrangements in which A and B will sit together.

Therefore, the number of arrangements in which they will not sit together = $6! - 2 \times 5!$

Now let's discuss some trickier variations of this question.

Question 3: 7 people, A, B, C, D, E, F and H, go to a movie and sit next to each other in 8 adjacent seats in the front row of the theatre. A and F will not sit next to each other in how many different arrangements?

Solution: Here, there is an additional vacant spot since there are only 7 people but 8 seats. You might think that it is a little confusing since you will need to deal with the vacant spot separately. Actually, this can be done in a very straight forward way.

7 people including A and F have to be seated such that A and F are not next to each other. So an arrangement where A and F have the vacant spot between them is acceptable. I will just imagine that there is an invisible person called Mr. V. He takes the vacant spot. If A and F have V between them, that arrangement is acceptable to us. Now this question is exactly like the question above. We have 8 people sitting in 8 distinct seats. 8 people (including our imaginary Mr. V) can sit in 8 seats in $8!$ arrangements.

A and F can sit together in $2 \times 7!$ arrangements (similar to question no. 2)

Hence, the number of arrangements in which A and F will not sit together = $8! - 2 \times 7!$

Quarter Wit_Quarter Wisdom- Part-1

Question 4: 7 people, A, B, C, D, E, F and H, go to a movie and sit next to each other in 8 adjacent seats in the front row of the theatre. In how many different arrangements will there be at least one person between A and F?

Solution: This variant wants you to put at least one person between A and F. This means that all those cases where A and F are together are not acceptable and all those cases where A and F have Mr. V (the vacant spot) between them are also not acceptable.

8 people (including our imaginary Mr. V) can sit in 8 seats in $8!$ ways.

A and F can sit together in $2*7!$ arrangements (similar to question no. 2). Number of arrangements in which A and F have Mr. V between them = $2*6!$

How? Now we group AVF together and consider this group one person. So there are 6 distinct individuals/groups which can be arranged in $6!$ ways. But we have 2 arrangements in this group: AVF and FVA. So total number of arrangements here = $2*6!$

These cases are not acceptable.

Hence, the number of arrangements in which A and F will have a person between them = $8! - 2*7! - 2*6! = 8! - 16*6!$

Compare question no. 3 with question no. 4: one where you don't want them to be together, the other where you don't want them to be together and you don't want the vacant spot between them.

Obviously, in the second case, the number of cases you do not want are higher. So you subtract a higher number out of the total number of cases.

Let me leave you with a question now. Make sure you answer exactly what is asked.

Question 5: 6 people go to a movie and sit next to each other in 6 adjacent seats in the front row of the theatre. If A cannot sit to the right of F, how many different arrangements are possible?

46.Linear Arrangement Constraints- II

Today, I want to discuss the symmetry principle of linear arrangements with you. If you do not understand the symmetry principle, then it is possible that the following has happened with you:

You see a hard question and start working on it. You know that there are going to be three-four different cases. You find the number of arrangements in each case. Then, very carefully, you add them all up and get your answer. You check the answer key and behold, your answer is correct. Just for the fun of it, you turn your page to the solutions section and see that there are just two lines there which go something like this: "You can arrange 6 people in $6!$ ways. In half of these $6!$ ways, A will be ahead of B so answer is $6!/2$." and you end up feeling pretty unhappy even though you got the correct answer!

To ensure that this doesn't happen again, let's try and understand the symmetry principle.

Let's work on a simple example first:

Question: There are 3 contestants, A, B and C. In how many different ways can they complete a race if the race doesn't end in a dead heat?

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Solution: Since the race doesn't end in a dead heat, there is no tie. The following arrangements are possible:

A B C

A C B

B A C

B C A

C A B

C B A

A total of $3! = 6$ arrangements. The first position is occupied by the contestant whose name is written first i.e. A B C implies A stands first, B stands second and C stands third in the race.

In how many of these arrangements is A ahead of B? We count and get 3 (A B C, A C B and C A B)

In how many of these arrangements is B ahead of A? We count and get 3 again (B A C, B C A, C B A)

The question is that out of 6 arrangements, why is it that in half A is ahead and in the other half, B is ahead? This is so because the arrangements are symmetrical. Each element has the same status. Since we are taking into account all arrangements, if half of them are partial toward A, other half have to be partial toward B. There is no difference between A and B. They are considered equal elements. Now if I ask you the number of arrangements in which B is ahead of C, you should jump up and say 3 immediately.

Let's now look at the question I left you with in the last post.

Question 6: 6 people go to a movie and sit next to each other in 6 adjacent seats in the front row of the theatre. If A cannot sit to the right of F, how many different arrangements are possible?

Solution: 'to the right of F' means anywhere on the right of F, not necessarily on the adjacent seat. Here we see symmetry because there are only 2 ways in which A can sit. In every arrangement, A is either to the left of F (any seat on the left) or to the right of F (any seat on the right). There is nothing else possible. The number of cases in which A will sit to the left of F will be the same as the number of cases in which he will sit to the right of F. That is why the answer here will be $6!/2 = 360$.

I hope you understand this principle now.

Let's quickly look at a couple of variants now.

Question 7: 7 people (A, B, C, D, E, F and G) go to a movie and sit next to each other in 7 adjacent seats in the front row of the theatre. A will not sit to the left of F and B will not sit to the left of E. How many different arrangements are possible?

Solution: Number of ways of arranging 7 people in 7 seats is $7!$ (using Basic Counting Principle)

Of these $7!$ arrangements, we want those arrangements in which A is sitting to the right of F and B is sitting to the right of E. A will sit to the right of F in half of the $7!$ arrangements. Of these $7!/2$ arrangements, half will have B to the right of E and other half will have B to the left of E. So the number of arrangements in which A is to the right of F and B is to the right of E is $(7!/2)/2 = 7!/4$

Question 8: 7 people (A, B, C, D, E, F and G) go to a movie and sit next to each other in 8 adjacent seats in the front row of the theatre. A will not sit to the left of F in how many different arrangements?

Solution: We have a vacant spot here. Recall the way we deal with vacant spots (discussed in the [last post](#)) — we use Mr V. 8 people (including our imaginary Mr V) can be arranged in 8 seats in $8!$ ways.

We want only those arrangements in which A is sitting to the right of F. In half of the $8!$ arrangements, A must be to the right of F (same as before) so required number of arrangements = $8!/2$

There are many more variations possible but I will stop here. Try some on your own and get back if you have a doubt. I will discuss some other little concept of Combinatorics with you next week.

47.Circular Arrangement constraints-I

In the last two posts, we discussed how to easily handle constraints in linear arrangements. Today we will discuss how to handle constraints in circular arrangements, which are actually even simpler to sort out. Let's look at some examples.

Quarter Wit_Quarter Wisdom- Part-1

Question 1: Seven people are to be seated at a round table. Andy and Bob don't want to sit next to each other. How many seating arrangements are possible?

Solution: There are 7 people who need to be seated around a circular table. Number of arrangements in which 7 people can be seated around a circular table = $(7-1)! = 6!$

(If you are not sure how we got this, check out [this post](#).)

We need to find the number of arrangements in which two of them do not sit together. Let us instead find the number of arrangements in which they will sit together. We will then subtract these arrangements from the total $6!$ arrangements.

Consider Andy and Bob to be one unit. Now we need to arrange 6 units around a round table. We can do this in $5!$ ways. But Andy and Bob can swap places so we need to multiply $5!$ by 2.

Number of arrangements in which Andy and Bob do sit next to each other = $2*5!$

So, number of arrangements in which Andy and Bob don't sit next to each other = $6! - 2*5!$

This is very similar to the way we handled such constraints in linear arrangements.

Question 2: There are 6 people, A, B, C, D, E and F. They have to sit around a circular table such that A cannot sit next to D and F at the same time. How many such arrangements are possible?

Solution: Total number of ways of arranging 6 people in a circle = $5! = 120$

Now, A cannot sit next to D and F simultaneously.

Let's first find the number of arrangements in which A sits between D and F. In how many of these 120 ways will A be between D and F? Let's consider that D, A and F form a single unit. We make DAF sit on any three consecutive seats in 1 way and make other 3 people sit in $3!$ ways (since the rest of the 3 seats are distinct). But D and F can swap places so the number of arrangements will actually be $2*3! = 12$

In all, we can make A sit next to D and F simultaneously in 12 ways.

The number of arrangements in which A is not next to D and F simultaneously is $120 - 12 = 108$.

A slight variation of this question that would change the answer markedly is the following:

Question 3: There are 6 people, A, B, C, D, E and F. They have to sit around a circular table such that A can sit neither next to D nor next to F. How many such arrangements are possible?

Solution: In the previous question, A could sit next to D and F; the only problem was that A could not sit next to both of them at the same time. Here, A can sit next to neither D nor F. Generally, it is difficult to wrap your head around what someone cannot do. It is easier to consider what someone can do and go from there. A cannot sit next to D and F so he will sit next to two of B, C and E.

Let's choose two out of B, C and E. In other words, let's drop one of B, C and E. We can drop one of B, C and E in 3 ways (we can drop B or C or E). This means, we can choose two out of B, C and E in 3 ways (We will come back to choosing 2 people out of 3 when we work on combinations). Now, we can arrange the two selected people around A in 2 ways (say we choose B and C. We could have BAC or CAB). We make these three sit on any three consecutive seats in 1 way.

Number of ways of choosing two of B, C and E and arranging the chosen two with A = $3*2 = 6$

The rest of the three people can sit in three distinct seats in $3! = 6$ ways

Total number of ways in which A will sit next to only B, C or E (which means A will sit neither next to D nor next to F) = $6*6 = 36$ ways

Now we will look at one last example.

Question 4: Six people are to be seated at a round table with seats arranged at equal distances. Andy and Bob don't want to sit directly opposite to each other. How many seating arrangements are possible?

Solution: Directly opposite means that Andy and Bob cannot sit at the endpoints of the diameter of the circular table.

Total number of arrangements around the circular table will be $(6-1)! = 5!$

But some of these are not acceptable since Andy sits opposite Bob in these. Let us see in how many cases Andy doesn't sit opposite Bob. Let's say we make Andy sit first. He can sit at the table in 1 way since all the seats are exactly identical for him.

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Now there are 5 seats left but Bob can take a seat in only 4 ways since he cannot occupy the seat directly opposite Andy.

Now there are 4 people left and 4 distinct seats left so they can be occupied in $4!$ ways.

Total number of ways of arranging the 6 people such that Andy does not sit directly opposite Bob = $1 \times 4 \times 4! = 96$ arrangements.

Make sure you understand the logic used in this question. We will build up on it in the next post.

48. Circular Arrangement constraints-II

With today's post, let's wrap up arrangements for the time being. We will discuss some complex circular arrangement constraints (which we will easily work through) today and start with combinations (i.e. picking "r" units out of "n" units) next week. Thereafter we will look at questions involving both, picking and arranging (yeah, that will be fun!).

Question 1: A group of 8 friends sit together in a circle. If A refuses to sit beside B unless C sits on the other side of A as well, how many possible seating arrangements are possible?

Solution: Let's start with what we know. We know that the total number of ways in which 8 people can be arranged around a circular table is $(8-1)! = 7!$

Since we do not want A to sit next to B, let's try and make them sit together. This will give us the number of arrangements that are unacceptable to us. Let's say that A and B are a single unit. So now there are 7 units which need to be arranged in a circle. This can be done in $(7-1)! = 6!$ ways. Since there are two arrangements possible, AB and BA, within the unit, we need to multiply $6!$ by 2.

Number of arrangements in which A and B sit together = $2 \times 6!$

We can subtract these 'unacceptable arrangements' from total arrangements to get the number of 'acceptable arrangements'. But this number of 'unacceptable arrangements' includes those arrangements where C is sitting on the other side of A. But those arrangements are acceptable to us so we should not subtract them out. How many such arrangements are there in which A and B are sitting together and C is sitting beside A too?

Now C, A and B form a single unit leaving us with 6 units to be arranged in a circle. 6 units can be arranged in $(6-1)! = 5!$ ways

CAB can also be arranged as BAC, hence the $5!$ needs to be multiplied by 2. (Mind you, we will not consider ABC, ACB etc here since A should be in the middle)

Number of arrangements in which A and B sit together and C sits beside A = $2 \times 5!$

Therefore, number of unacceptable arrangements = $2 \times 6! - 2 \times 5!$

We subtract these out of the total number of arrangements and we get the total number of acceptable arrangements.

Possible number of seating arrangements = $7! - (2 \times 6! - 2 \times 5!) = 3840$

If you are wondering about the 'painful' calculation involved in the step above, don't worry. Calculations with factorials are generally quite straight forward.

$$7! - (2 \times 6! - 2 \times 5!) = 7! - 2 \times 6! + 2 \times 5!$$

$$= 2 \times 5! (21 - 6 + 1) \text{ (Take } 2 \times 5! \text{ common out of the three terms)}$$

$$= 2 \times 120 \times 16 = 32 \times 120 = 3840$$

I hope the solution makes sense to you. Let's look at another tricky circular arrangement problem.

Quarter Wit_Quarter Wisdom- Part-1

Question 2: Seven men and seven women have to sit around a circular table so that no two women are together. In how many different ways can this be done?

Solution:

There are 7 men: Mr. A, Mr. B, Mr. C, Mr. D ...

and 7 women: Ms. A, Ms. B, Ms. C, Ms. D ...

Let's say we have 14 identical chairs around the round table.

We need to seat the 7 women such that no two of them are together i.e. there should be a man on either side of every woman. Since there are exactly 7 men, the women and men should sit alternately. Let's make the women sit first. For the first woman who sits, each seat is identical so she sits in one way (say Ms.C takes a seat). Now each seat is distinct relative to this woman (Ms. C). There are 7 seats identified for men (e.g. seats right next to Ms. C and every alternate seat) and 6 for the remaining 6 women. The 7 men can occupy the 7 distinct seats in $7!$ ways and the 6 women can occupy the 6 distinct seats in $6!$ ways.

Total number of arrangements = $6! \cdot 7!$

Something to ponder upon: The total number of arrangements is not $13!$. Why?

Question 3: Find the number of ways in which four men, two women and a child can sit at a circular table if the child is seated between the two women.

Solution:

We have 7 people and 7 seats around a circular table.

First let's make the child sit anywhere in one way since all the places are identical. The two women can sit around the child in $2!$ ways. Now we have 4 distinct seats (relative to the people sitting) left for the 4 men and they can occupy the seats in $4!$ ways.

Total number of arrangements = $1 \cdot 2! \cdot 4! = 48$

Things to ponder upon:

Case 1: Same question as above but the chairs are numbered i.e. all the seats are distinct. Find the number of ways in which four men, two women and a child can sit around a circular table with numbered seats if the child is seated between the two women.

Case 2: Same question as above but they need to stand in a row instead. Find the number of ways in which four men, two women and a child can stand in a row if the child is standing between the two women.

Are the two cases above equivalent?

49.Considering Combinations

We will start with Combinations today. The moment we start talking about Permutations and Combinations, the first question many people ask is: "How do I know whether the given problem is a combinations problem or a permutations problem?"

My answer is: "Focus on what you have to do. Do you have to just SELECT some friends/toys/candies/candidates etc or do you have to ARRANGE them in distinct seats/among some children/in distinct positions etc too. If you have to only select, it is a combinations problem; if you have to only arrange, it is a permutations problem; if you have to first select and then arrange, it is a combinations and permutations problem. But if you are not using the formulas (nPr and nCr), you don't have to think in terms of permutations and combinations. Just think in terms of selecting and arranging." In the

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discussion below, I will start with an explanation of how we can make selections and how we can work on combinations without using the formula. We will also take a quick look at the formula and why it is what it is. Then we will move on to some examples.

I hope you remember the [basic counting principle](#) that we looked at some weeks back. We can use the same to understand combinations too. Let's see how.

Say, there are 5 friends but only 3 seats in a row. In how many ways can you make 3 of the 5 friends sit in the 3 seats?

We start by using the basic counting principle.

We have 3 seats ____ ____ ____

In how many ways can we make someone sit on the leftmost seat? In 5 ways. In how many ways can we make someone sit on the middle seat? In 4 ways. In how many ways can we make someone sit on the rightmost seat? In 3 ways. Then in how many ways can we fill all the 3 seats? In $5 \times 4 \times 3 = 60$ ways.

Here, we have effectively selected 3 people out of 5 and arranged them in 3 seats. What if we had to only select and not arrange?

Say you have 5 friends and you have to invite any 3 of them to go with you on a vacation. In how many ways can you do that?

Will the answer still be 60? No because 60 includes the different arrangements too. In this case, we only need to select 3 friends. We don't have to arrange them in 3 positions. What do you do if you want to un-arrange 3 people? You arrange 3 people by multiplying by $3!$. Therefore, you can un-arrange 3 people by dividing by $3!$.

Number of ways of selecting 3 people out of 5 = $60/3! = 10$ ways

This is equivalent to using the formula:

Number of ways of selecting r people out of a total of n people = ${}^nC_r = n!/(r! * (n-r)!)$

Number of ways of selecting 3 people out of a total of 5 people = ${}^5C_3 = 5!/(3! * (5-3)!) = 10$

I hope you understand the logic behind the formula. If you don't want to use the formula, don't. You can just think in terms of basic counting principle and un-arranging. Let's look at a couple of examples now.

Question 1: A company consists of 5 senior and 3 junior staff officers. If a committee is created with 3 senior and 1 junior staff officers, in how many ways can the committee be formed?

- (A) 12
- (B) 30
- (C) 45
- (D) 80
- (E) 200

Solution:

You have to select 3 senior and 1 junior officers. Note here that you don't have to arrange them in any way. You just have to select.

There are a total of 5 senior officers. You can select 3 of them in $5 \times 4 \times 3/3!$ ways. Note that we divide by $3!$ to un-arrange.

There are 3 junior officers and you have to select one of them. You can do that in 3 different ways. Note here that you don't need to do any calculations when you have to select just one person. Out of 3 people (say A, B and C), you can select one in 3 ways (you can select A or B or C).

Quarter Wit_Quarter Wisdom- Part-1

So you can select 3 senior and 1 junior officers in $5 \cdot 4 \cdot 3 / 3! \cdot 3 = 30$ ways

Answer (B)

Question 2: A class is divided into four groups of four students each. If a project is to be assigned to a team of three students, none of which can be from the same group, what is the greatest number of distinct teams to which the project could be assigned?

- (A) 4^3
- (B) 4^4
- (C) 4^5
- (D) $6(4^4)$
- (E) $4(3^6)$

Solution: We need to make a team here. There is no arrangement involved so it is a combinations problem. First we will select 3 groups and then we will select one student from each of those 3 groups.

In how many ways can we select 3 groups out of a total of 4? From the theory discussed above, I hope you agree that we can select 3 groups out of 4 in $4 \cdot 3 \cdot 2 / 3! = 4$ ways. The interesting thing to note here is that selecting 3 groups out of 4 is the same as selecting 1 group out of 4. Why? Because we can think of making the selection in two ways – we can select 3 groups from which we will pick a student each or we can select 1 group from which we will not select a student. This will automatically give us a selection of 3 groups. We know that we can select 1 out of 4 in 4 ways (hence the calculation done above was actually not needed).

Now from each of the 3 selected groups, we have to pick one student. In how many ways can we select one student out of 4? In 4 ways. This is true for each of the three groups. We can select 3 groups and one student from each one of the three groups in $4 \cdot 4 \cdot 4 \cdot 4 = 4^4$ ways.

Answer (B)

Now that we have discussed the basic theory of combinations, next week we will discuss some combinations questions with constraints.

50. Combinations with Constraints

Last week, we discussed the basics of combinations. Until and unless you have worked a decent bit with combinatorics in high school, the formula of combinations will not be very intuitive. We have already discussed how you can easily think of “selection” in terms of basic counting principle and un-arranging instead of the formula, if you so desire. Today, I would like to discuss some combination questions with constraints. A very common type of such questions asks you to make a committee of r people out of n people under some constraints. Let me show you what I mean with the help of some examples.

Question 1: If a committee of 3 people is to be selected from among 6 married couples such that the committee does not include two people who are married to each other, how many such committees are possible?

- (A) 20
- (B) 40
- (C) 80
- (D) 120
- (E) 160

Solution: We have 6 married couples i.e. we have 12 people. Out of these 12 people, we have to choose 3. If there were no

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constraints on the choice and we could choose any 3 out of these 12, in how many ways could we do that?

12 people and 3 available positions – We use basic counting principle to arrive at $12 \times 11 \times 10$. But, recall that we have arranged the 3 people here. To un-arrange, we divide this by $3!$ to get $12 \times 11 \times 10 / 3!$ arrangements. This is what we discussed last week.

Moving forward, this question has a constraint – no two people married to each other should be selected i.e. at most one of any two married people could be selected. Say, if we select Mr. X, we cannot select Ms. X and vice versa.

Let's try to use our basic counting principle and un-arranging method here:

You have 12 people and 3 positions. In the first position, you can put any one of the 12 people. In the second position, you can put any one of 10 people (not 11 because the spouse of the person put in 1st position cannot be put in the second position). In the third position you can put any one of 8 people. (We have already selected 2 people for the previous two places and their spouses cannot be selected for the third position so 4 of the 12 people are out.)

Total number of arrangements = $12 \times 10 \times 8$

But mind you, these are arrangements. We just need a group of 3 people. They don't need to be arranged in the group. So we divide these arrangements by $3!$ to just 'select' the people.

Number of committees possible = $12 \times 10 \times 8 / 3! = 160$

I think this was easy, right? Let's look at a little more complex problem now.

Question 2: A group of 10 people consists of 2 married couples and 6 bachelors. A committee of 4 is to be selected from the 10 people. How many different committees can be formed if the committee can consist of at most one married couple?

Solution: We have to select 4 people out of: 6 bachelors and 2 married couples.

The number of ways of selecting any 4 people out of 10 is $10 \times 9 \times 8 \times 7 / 4! = 210$ (Note here that we are just selecting 4 people. We are not arranging them so we divide by $4!$)

The people will get selected in various ways:

1. Four bachelors
2. One from a couple and three bachelors
3. Two from two different couples and two bachelors
4. One couple and two bachelors
5. One couple, one person from a couple, one bachelor
6. Two couples

If we add the number of committees possible in each of these cases, we will get 210. Out of all these cases, only the last one (two couples) has more than one married couple. Instead of calculating the number of different committees that can be formed in each of the first five cases, we can calculate the number of committees in the last case and subtract it from 210.

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How many different committees can be formed such that there are 2 couples? Only one since we have only 2 couples. We will have to select both the couples and we will get 4 people.

Number of different committees of 4 people such that there is at most one married couple = $210 - 1 = 209$.

Just for practice, let's see how we can calculate the different number of committees that can be formed in each of the first five cases. The sum of all these cases should give us 209.

1. Select 4 bachelors from 6 bachelors in $6 \cdot 5 \cdot 4 \cdot 3 / 4! = 15$ different committees
2. Select 1 person out of the two couples (4 people) in 4 ways and 3 bachelors from 6 bachelors in $6 \cdot 5 \cdot 4 / 3! = 20$ ways. So you select the 4 people in $4 \cdot 20 = 80$ different committees
3. Select 2 people from 2 different couples in $4 \cdot 2 / 2! = 4$ ways and 2 bachelors from 6 bachelors in $6 \cdot 5 / 2! = 15$ ways. So you select the 4 people in $4 \cdot 15 = 60$ different committees
4. Select 1 couple in 2 ways and 2 bachelors from 6 bachelors in $6 \cdot 5 / 2! = 15$ ways. So you select the 4 people in $2 \cdot 15 = 30$ different committees
5. Select 1 couple in 2 ways, 1 person from the remaining couple in 2 ways and 1 bachelor from 6 bachelors in 6 ways. So you can select the 4 people in $2 \cdot 2 \cdot 6 = 24$ different committees

The sum of all these five cases = $15 + 80 + 60 + 30 + 24 = 209$ different committees (as expected)

Let me add here that combinatorics is a huge topic. We can put up a 100 posts on it and still not exhaust all the content. If there is a particular concept that you would like me to discuss, let me know. Else, I will follow my own train of thought and discuss the most frequently occurring topics.